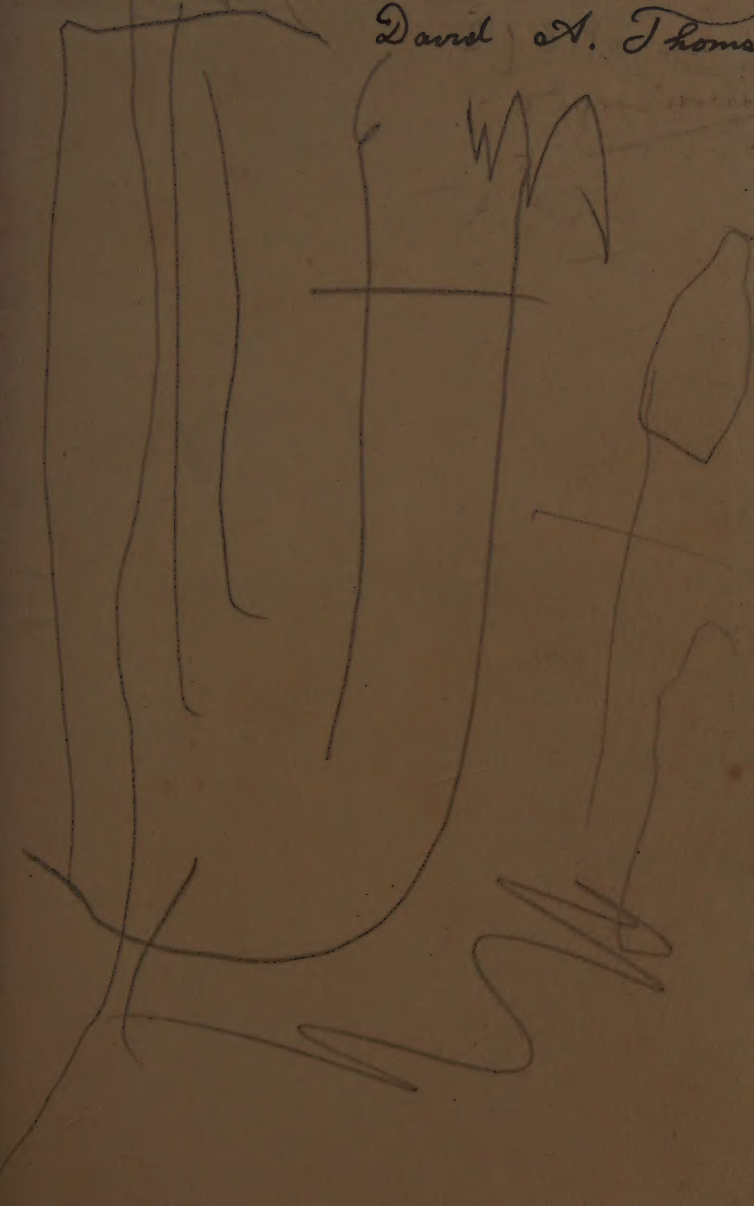
David A. Thoms

HIGHER GEOMETRY

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CONTAINING AN INTRODUCTION TO MODERN GEOMETRY AND ELEMENTARY GEOMETRICAL CONICS

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PREFACE

This short course of Modern Geometry and Geometrical Conics has been used in my own classes for a considerable number of years. It has been re-arranged from time to time, until it has acquired the form in which it is now printed; and it therefore represents the result of my experience as to the best method of introducing these subjects to boys of fifteen or sixteen, and the extent to which they may be carried in an ordinary school course. This latter consideration will account for the omission of some of the properties of the ellipse and hyperbola usually given in works which profess to treat these curves completely.

The sources of information of which I have availed myself will be obvious to any one who is acquainted with the literature of the subject, and I have given references whenever I had any confidence that I was quoting the original author. The works to which I feel myself most indebted, besides the original works referred to in the text, are Rouché et Comberousse, *Traité de Géométrie* (Appendices); Housel, *Introduction à la Géométrie Supérieure;* Taylor, *Ancient and Modern Geometry of Conics;* and the *Syllabuses* of the Association for the Improvement of Geometrical Teaching.

It is assumed that the reader knows the first six books of Euclid and the Elements of Trigonometry. References such as VI. 4 are to any ordinary edition of Euclid's Elements.

I have to thank my friend Dr J. S. Mackay for many of the historical references, as well as for his valuable assistance in reading the proofs.

W. J. MACDONALD.

EDINBURGH, 1890.

PREFACE TO SECOND EDITION

ONLY a very few alterations have been made on the text of the first edition. Some questions set in the Leaving Certificate Examinations have been added; and to the Conic Sections the propositions on the Section of the Cone.

W. J. M.

EDINBURGH, 1897.

PREFACE TO THIRD EDITION

IN this edition the questions set in the Leaving Certificate Examinations have been brought up to date, and a few misprints corrected.

W. J. M.

EDINBURGH, 1903.

MODERN GEOMETRY

INTRODUCTION

MODERN GEOMETRY differs from the Ancient or Greek Geometry, of which it is a natural development, chiefly in the recognition and application of two great principles—the *principle of continuity* and the *principle of duality*.

Principle of Continuity.—The Greek geometer treated the various cases of a theorem separately, and seems never to have looked for any general principle underlying and connecting them all. Who, for example, would ever suppose, from Euclid's treatment of them, that there was any very close connection between II. 4 and II. 7? On the other hand, the modern geometer seeks to group as many cases as possible under one general enunciation, bringing in the more important particular cases as a series of corollaries. The former considered each new arrangement of the elements of a figure as furnishing him with a new theorem. The latter, while he considers certain elements to remain constant, tries so to enunciate his theorem as to admit of a continuous variation of some of the other elements. Take, for instance, this enunciation: "The square on one side of a triangle differs from the squares on the other two sides by twice the rectangle

contained by either of these sides and the projection of the other upon it." A careful examination will show that, if we consider the two last-mentioned sides to remain constant in length while the first side and the angle opposite it vary, this enunciation contains Euclid II. 7, 13; I. 47; II. 12 and 4, according as the angle just mentioned is zero, acute, right, obtuse, or two right angles.

Let us consider a given figure in the most general form it can assume without violating certain conditions imposed upon it, and suppose that by the application of ordinary reasoning we have deduced under these conditions certain theorems relative to it. If now we cause the figure to vary continuously, while still remaining subject to the aforesaid conditions, the principle of continuity leads us to assert the truth of these theorems in every special form the figure can assume, due allowance being made for such magnitudes as may vanish, change sign, etc. (*Cf.* Poncelet, *Propriétés Projectives*, 2nd edition, p. xiii.) We are thus led to the consideration of limiting cases, such as the straight line and the point treated as infinitely great and infinitely small circles, parallels treated as straight lines which intersect at infinity, etc. Many illustrations of the use of this principle will be found in the following pages, especially among the corollaries and exercises.

Principle of Duality.—If one figure be derived from another, according to a definite law, there may be discovered rules by which theorems applicable to one of the figures may be derived from theorems which have been

shown to be true with respect to the other. The methods of *Similitude* and *Inversion* are methods by which one figure can be transformed into another, such that each point in a given figure corresponds to a point in the figure into which it has been transformed, and the method of *Reciprocal Polars* is one which gives a dual correspondence between points and straight lines. Notice also a similar correspondence in the sections on *Ranges and Pencils*, *Harmonic Ranges and Pencils*, and *Cross Ratios of Ranges and Pencils*, which I have endeavoured to bring out as clearly as possible by wording each pair of definitions or of theorems so that there is scarcely any difference between them except that the expression *point* and *straight line* (or *ray*) are interchanged.

The foregoing does not profess to be anything like a complete exposition of the subject, but I hope it will be found accurate as far as it goes, and sufficient to enable the reader to approach the subject from the right point of view.

[We find the principle of continuity first recognised (though not by that name) and applied in the geometrical writings of *Kepler* (1571-1631), *Desargues* (see Th. 4), *Pascal* (see Th. 5), and a few of their contemporaries and immediate successors.

During the eighteenth century the energies of all the greatest mathematicians were concentrated on the development of the differential and integral calculus.

The beginning of a new period in the history of pure

geometry is marked by the publication of the *Géométrie Descriptive* of Monge (1800), and of the *Essai sur la Théorie des Transversales* of Carnot (1806). In 1813-14, when a prisoner in Russia, J. V. Poncelet, a pupil of Monge, worked out a system of geometry founded on the principles of continuity and duality, which he published in 1822, under the title, *Traité des Propriétés Projectives des Figures* (2nd edition, 1865), and which may be considered as the first treatise on modern geometry. In his *Systematische Entwickelung*, etc. (1832), Jacob Steiner developed in great detail the results of the principle of duality.

The great importance of the cross ratio of a range or pencil was first recognised by Michel Chasles (1793-1880), and its consequences followed out in his classical works, *Traité de Géométrie Supérieure* (1852) and *Traité des Sections Coniques* (1865).]

MODERN GEOMETRY.

RANGES AND PENCILS.

TRANSVERSALS.

Def. A system of points in a straight line is called a *range:* the line in which they lie is called the *base* of the range: the points are said to be *collinear.*

Two points intercept a *segment.*

Segments measured in one direction along the same or parallel straight lines are reckoned positive (+), and in the opposite direction negative (−).

+AB may be interpreted "go from A to B," and −AB "go from B to A."

Hence, $AB = -BA$;

$$AB + BC + CA = 0;$$

$$AB - CB = AB + BC = AC;$$

A, B, and C being any three collinear points.

Ex. 1. AA_1BB_1 is a range of four points, α is the middle point of AA_1 and β that of BB_1: prove

i. $AB + BA_1 + A_1B_1 + B_1A = 0.$

ii. $AB^2 - A_1B_1{}^2 = (AB_1 - BA_1)\ (AA_1 - BB_1).$

iii. $AB^2 + A_1B_1{}^2 = AB_1{}^2 + BA_1{}^2 - 2AA_1 \,.\, BB_1.$

iv. $A\alpha^2 - B\alpha \,.\, B_1\alpha = B\beta^2 - A\beta \,.\, A_1\beta.$

v. $AB \,.\, AB_1 - A_1B \,.\, A_1B_1 = A\beta^2 - A_1\beta^2.$

Ex. 2. A, B, C, D are four collinear points: show that $AB \,.\, CD + BC \,.\, AD + CA \,.\, BD = 0.$ Connect this with the identity $(a-b)(c-d)+(b-c)(a-d)+(c-a)(b-d)=0,$ and with Ptolemy's Theorem.

[This theorem was first given by Euler in 1747: *Novi Commentarii Academiae Petropolitanae*, vol. i. p. 49.]

Ex. 3. A, B, and C are fixed points, and M a variable point in a range: show that

i. $AM^2 + BM^2 + CM^2 - MA \,.\, MB - MA \,.\, MC - MB \,.\, MC$
$= AB \,.\, AC + BA \,.\, BC + CA \,.\, CB.$

ii. $AM^2 \,.\, BC + BM^2 \,.\, CA + CM^2 \,.\, AB + AB \,.\, BC \,.\, CA = 0.$

Def. A system of straight lines (*rays*) passing through a point is called a *pencil*: the point through which they pass is called the *base* or *vertex* of the pencil: the rays are said to be *concurrent*.

Two rays intercept an *angle*.

Angles measured in one direction round a point are reckoned positive (+), and in the opposite direction negative (−).

$+ \angle AOB$ may be interpreted "turn round O from A to B," and $- \angle AOB$ "turn round O from B to A."

Hence, $\angle AOB = - \angle BOA$;

$\angle AOB + \angle BOC + \angle COA = 0$;

$\angle AOB - \angle COB = \angle AOB + \angle BOC = \angle AOC$;

OA, OB, and OC being any three concurrent straight lines.

In the above and in all that follows, the rays are understood to lie in the same plane or to form a *flat pencil.*

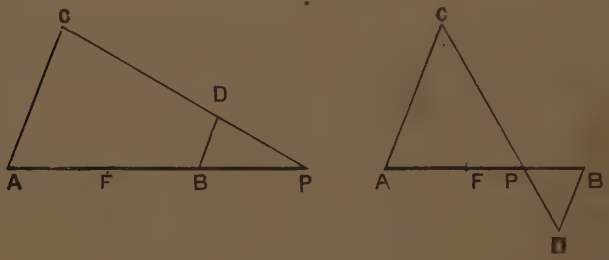
Def. If in a range two points be fixed (*base points*), the position of any other point in the range is determined if the ratio of its distances from the base points is known (Prob. 1, Cor. 1).

This ratio is called the *position ratio* of the point.

If A and B are the base points, we shall always write the position ratio of P (any other point) AP : BP, thus making the position ratio positive (AP and BP being of the same sign) when P divides AB externally, and negative (AP and BP being of opposite sign) when P divides AB internally.

PROBLEM 1.

To construct a point in a range, having given the base points and its position ratio.



Let A and B be the base points and $\pm \frac{m}{n}$ the given ratio.

Through A draw any straight line AC m units of length, and through B draw BD parallel to AC and n units of length, drawing AC and BD in the same or opposite directions according as the given ratio is positive or negative.

Join CD meeting AB in P:

P is the required point.

$\triangle$s ACP, BDP are similar, *VI.* 4.

$$\therefore \frac{AP}{BP} = \frac{AC}{BD} = \pm \frac{m}{n}.$$

Cor. I. A straight line can be divided in any given ratio in one point only, if the sign as well as the numerical value of the ratio be taken into account.

Cor. 2. Let F be the middle point of AB.

i. When $m : n$ is positive and $m < n$, P lies in BA produced.

ii. „ „ „ and $m > n$, „ AB „

iii. „ „ negative and $m < n$, „ AF.

iv. „ „ „ and $m > n$, „ FB.

v. „ $m : n = 0$, P is at A.

vi. „ „ $= \infty$, P „ B.

vii. „ „ $= -1$, P „ F.

viii. „ „ $= +1$, P „ ∞.

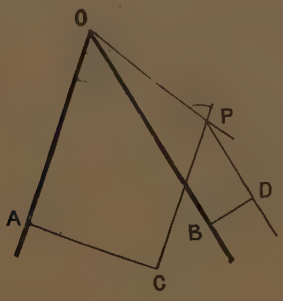
Def. If in a pencil two rays be fixed (*base rays*), the position of any other ray in the pencil is determined if the ratio of the *sines* of its angular distances from the base rays is known (Prob. 2, Cor. 1).

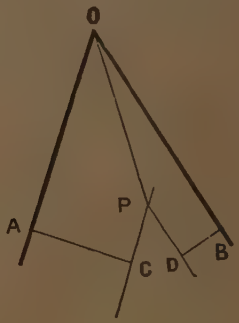
This ratio is called the *position ratio* of the ray.

If OA and OB are the base rays, we shall always write the position ratio of OP (any other ray) sin AOP : sin BOP, thus making the position ratio positive or negative according as OP lies in the supplement of $\angle$ AOB or in $\angle$ AOB itself.

PROBLEM 2.

To construct a ray in a pencil, having given the base rays and its position ratio.





Let OA and OB be the base rays and $\pm \frac{m}{n}$ the given ratio.

At any points A and B in the base rays draw AC and BD perpendicular to OA and OB and m and n units of length respectively—drawing them in the same or opposite (cyclical) directions, according as the given ratio is positive or negative.

Through C and D draw parallels to OA and OB, meeting in P.

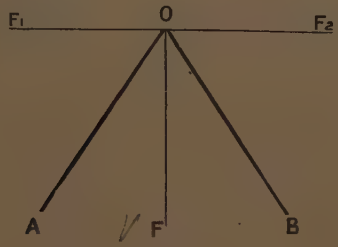
Join OP.

OP is the required ray.

For $\frac{\sin AOP}{\sin BOP} = \frac{AC}{OP} \div \frac{BD}{OP} = \frac{AC}{BD} = \pm \frac{m}{n}$.

Cor. 1. An angle can be divided into two parts, the ratio of the sines of which are in any given ratio by one line only, if the sign as well as the numerical value of the ratio be taken into account.

Cor. 2. Let OF bisect the angle AOB and F_1OF_2 its supplement.



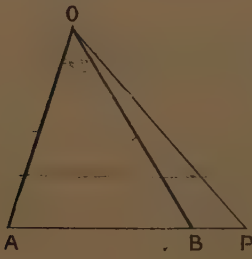
i. When $m : n$ is positive and $m < n$, OP lies in $\angle AOF_1$.
ii. „ „ „ and $m > n$, „ „ $\angle BOF_2$.
iii. „ „ negative and $m < n$, „ „ $\angle AOF$.
iv. „ „ „ and $m > n$, „ „ $\angle BOF$.
v. „ $m : n = 0$, OP coincides with OA.
vi. „ $m : n = \infty$, „ „ „ OB.
vii. „ $m : n = -1$, „ „ „ OF.
viii. „ $m : n = +1$, „ „ „ OF_1 or OF_2.

Def. A straight line drawn across a system of other lines is called a *transversal.*

THEOREM 1.

If a transversal meet the base rays and any third ray of a pencil whose vertex is O *in* A, B, *and* P *respectively,*

$$\frac{AP}{BP} = \frac{OA}{OB} \cdot \frac{\sin AOP}{\sin BOP}.$$



$$\frac{AP}{BP} = \frac{\triangle AOP}{\triangle BOP}, \qquad VI.\ 1.$$

$$= \frac{\frac{1}{2} OA \cdot OP \cdot \sin AOP}{\frac{1}{2} OB \cdot OP \cdot \sin BOP},$$

$$= \frac{OA}{OB} \cdot \frac{\sin AOP}{\sin BOP}.$$

Cor. 1. If angles AOP and BOP are equal or supplementary, the theorem becomes VI. 3, or A.

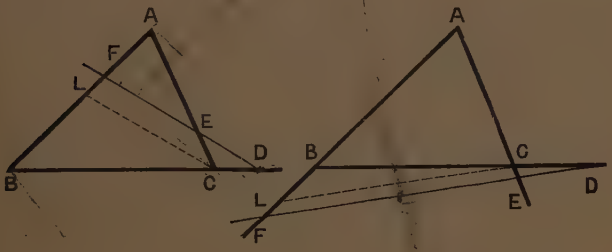
Cor. 2. If OA = OB, the position ratio of P with respect to A and B is equal to that of OP with respect to OA and OB.

Cor. 3. Parallel transversals are cut proportionally by the rays of a pencil.

THEOREM 2.

If a transversal cut the sides a, b, and c of a triangle ABC *in* D, E, *and* F *respectively,* $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1.$

Conversely : If D, E, *and* F *be points in the sides a, b, and c respectively of a triangle* ABC *such that* $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = 1,$ D, E, *and* F *are collinear.*



Draw CL parallel to DEF. *I.* 31.

$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{BF}{LF} \cdot \frac{LF}{AF} \cdot \frac{AF}{BF} = +1.$ *VI.* 2.

[The product is positive because the ratios are either all positive, or two negative and one positive.]

Converse : Suppose that DE meets AB in F_1 (not shown in the fig.).

$\triangle$ ABC is cut by the transversal DEF_1

$$\therefore \quad \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF_1}{BF_1} = 1$$

$$= \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF}; \qquad \textit{Hyp.}$$

$$\therefore \quad \frac{AF_1}{BF_1} = \frac{AF}{BF}$$

$\therefore$ F and F_1 coincide, *Prob.* 1, *Cor.* 1.

i.e., D, E, and F are collinear.

[This theorem is frequently called "Menelaus' Theorem," from Menelaus, an Alexandrian mathematician, who flourished about 80 A.D., and in whose treatise on *Spherics* it is first found. See Chasles, *Aperçu Historique &c.*, note vi.]

Ex. 4. Prove Menelaus' Theorem by projecting the vertices of the triangle

(i.) On any straight line perpendicular to the transversal and using VI. 2;

(ii.) On the transversal and using VI. 4.

Ex. 5. If in the figure of Theorem 2, AD, BE, and CF be joined, prove that

$$\frac{\sin BAD}{\sin CAD} \cdot \frac{\sin ACF}{\sin BCF} \cdot \frac{\sin CBE}{\sin ABE} = 1.$$

Ex. 6. Extend Menelaus' Theorem to the case of any polygon cut by a transversal. (Project the vertices on a straight line perpendicular to the transversal, and use VI. 2.). (Carnot, *Essai*, p. 70.)

Ex. 7. Two internal and the third external angle bisectors of a triangle meet the opposite sides in three collinear points.

Ex. 8. The external angle bisectors of a triangle meet the opposite sides in three collinear points.

Ex. 9. Tangents to a circle at the vertices of an inscribed triangle meet the opposite sides of the triangle in three collinear points.

Ex. 10. Perpendiculars from any point on the circumference of a circle to the sides of an inscribed triangle meet these sides in three collinear points.

[First given by W. Wallace, in Leybourn's *Math. Repository*, Old Series, vol. ii., p. 111. Often and erroneously attributed to Simson.]

Ex. 11. Show that VI. 2, 3, and A are limiting cases of Menelaus' Theorem.

Ex. 12. If a circle cut the sides of a triangle, or those sides produced, a in D and D_1, b in E and E_1, and c in F and F_1, show that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{CD_1}{BD_1} \cdot \frac{AE_1}{CE_1} \cdot \frac{BF_1}{AF_1}.$$

(Carnot, *Essai*, p. 72.)

Ex. 13. Show that Menelaus' Theorem is a limiting case of Ex. 12. (Consider the circle to become infinitely great, *i.e.*, a straight line, and apply Prob. 1, Cor. 2, viii.)

Ex. 14. Extend Ex. 12 to any polygon cut by a circle. (Carnot, *Essai*, p. 72.)

Ex. 15. Replace the circle in Ex. 12 and 14 by a pair of straight lines.

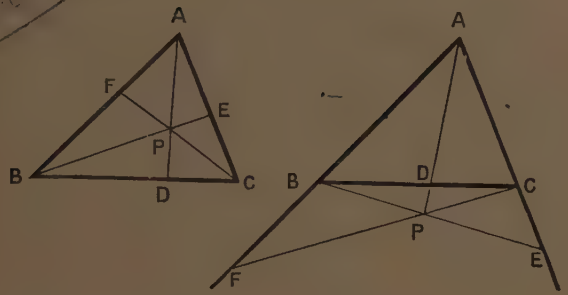
Ex. 16. ABC is any triangle, a circle whose centre is A cuts AB and AC in E and F respectively, the median AD cuts the chord EF in G : show that $\frac{EG}{FG} = \frac{AC}{AB}$.

Ex. 17. Through any point P in the plane of a triangle ABC are drawn AD, BE, and CF cutting the opposite sides in D, E, and F respectively : show that $\frac{AP}{DP} = \frac{AE}{CE} + \frac{AF}{BF}$.

THEOREM 3.

If through any point P *in the plane of a triangle* ABC *straight lines* AP, BP, *and* CP *be drawn meeting the opposite sides in* D, E, *and* F *respectively,* $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1$.

Conversely : If D, E, *and* F *be three points in the sides of a triangle* ABC *such that* $\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1$, AD, BE, *and* CF *are concurrent.*



△ ABD is cut by the transversal CPF;

$$\therefore \frac{BC}{DC} \cdot \frac{DP}{AP} \cdot \frac{AF}{BF} = 1. \qquad \textit{Th. 2.}$$

△ ADC is cut by the transversal BPE;

$$\therefore \frac{DB}{CB} \cdot \frac{CE}{AE} \cdot \frac{AP}{DP} = 1. \qquad \textit{Th. 2.}$$

Multiply these results, and attend to the signs of the segments;

$$\therefore \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = -1.$$

Converse: Let AD and BE meet in P.

Join CP meeting AB in F_1 (not shown in fig.).

AD, BE, and CF_1 are concurrent,

$$\therefore \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF_1}{BF_1} = -1$$

$$= \frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF}; \qquad \textit{Hyp.}$$

$$\therefore \frac{AF_1}{BF_1} = \frac{AF}{BF},$$

$\therefore$ F and F_1 coincide, *Prob.* 1, *Cor.* 1.

i.e., AD, BE, and CF are concurrent.

[This theorem is frequently called "Ceva's Theorem," from Giovanni Ceva, in whose work, *De lineis rectis se invicem secantibus, statica constructio* (Milan, 1678), it first occurs. See Chasles, *Aperçu*, note vii.]

The above is the usual way of presenting Theorems 2 and 3, but the true relation of the two theorems will be better seen if they are enunciated as follows:

Th. 2. *If in the sides of a triangle there be taken three collinear points, one in each side, the product of their position ratios with respect to the vertices of the triangle is* $+1$: *and conversely.*

Th. 3. *If through the vertices of a triangle there be drawn three concurrent straight lines, one through each vertex, the product of their position ratios with respect to the sides of the triangle is* -1 : *and conversely.*

Proof of Theorem 2, as given above.

Proof of Theorem 3 (see figure of Theorem 3 above).

$$\frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBP}{\sin ABP} \cdot \frac{\sin ACP}{\sin BCP}$$

$$= \frac{\sin BAP}{\sin ABP} \cdot \frac{\sin CBP}{\sin BCP} \cdot \frac{\sin ACP}{\sin CAP}$$

$$= \frac{BP}{AP} \cdot \frac{CP}{BP} \cdot \frac{AP}{CP} = -1.$$

[The product is negative because the position ratios are either all negative, or two positive and one negative.]

The proof of the converse is similar to that previously given.

In using Ceva's Theorem it will sometimes be found better to employ the first form of the enunciation, and sometimes the second.

In the following exercises, the form which should be used is indicated.

Ex. 18. Prove Ceva's Theorem (first form) by applying this theorem : "Triangles on the same base are to each other as the segments into which the straight line joining their vertices is divided by the base."

Ex. 19. Extend Ceva's Theorem (first form) to polygons with an odd number of sides.

Ex. 20. Extend Ceva's Theorem (second form) to any polygon (taking ± 1 according as the number of sides is even or odd).

Ex. 21. The medians of a triangle are concurrent. (Use first form.) (Archimedes, *De Planorum Æquilibriis*, i., 13, 14.)

Ex. 22. The straight lines which join the vertices of a triangle with the points of contact of the inscribed circle are concurrent. (Use first form.)

Ex. 23. The straight lines which join the vertices of a triangle with the points of contact of an escribed circle are concurrent. (Use first form.)

Ex. 24. The internal angle bisectors of a triangle are concurrent. (Use second form.)

Ex. 25. One internal, and the other two external angle bisectors of a triangle are concurrent. (Use second form.)

Ex. 26. The altitudes of a triangle are concurrent. (Use second form.) (Archimedes, *Lemmas*, v.)

Def. Two straight lines are said to be *isogonal conjugates* with respect to an angle, when they are equally inclined to its bisector.

Def. The isogonal conjugates of the medians of a triangle with respect to the angles through which they pass, are called its *symmedians*.

Ex. 27. The symmedians of a triangle are concurrent. (Use second form.)

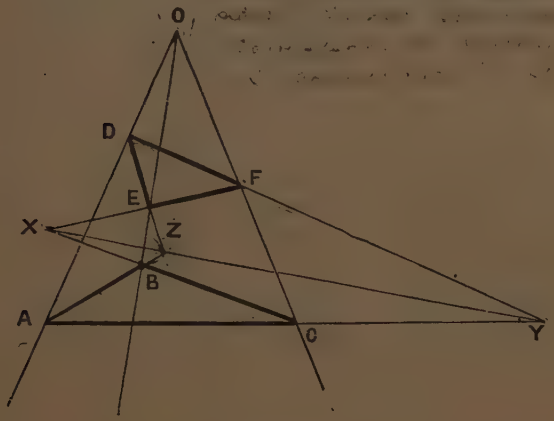
Ex. 28. If three straight lines through the angles of a triangle be concurrent, their isogonal conjugates are also concurrent. (Use second form.)

(Steiner, *Gesammelte Werke*, vol. i., p. 193).

Ex. 29. If a circle cut the sides of a triangle or those sides produced, a in D and D_1, b in E and E_1, and c in F and F_1, and if AD, BE, and CF are concurrent, so are AD_1, BE_1, and CF_1. (See Ex. 12.)

THEOREM 4.

If the corresponding vertices of two triangles lie on three concurrent straight lines, the points of intersection of their corresponding sides are collinear; and conversely.



Let $\triangle$s ABC and DEF have their corresponding vertices on the concurrent straight lines OA, OB, and OC: it is required to prove that the points of intersection of their corresponding sides, viz., X, Y, and Z, are collinear.

$\triangle$OAB is cut by the transversal DEZ,

$$\therefore \frac{AZ}{BZ} \cdot \frac{BE}{OE} \cdot \frac{OD}{AD} = 1. \qquad \textit{Th. 2.}$$

$\triangle$ OBC is cut by the transversal FEX,

$$\therefore \frac{BX}{CX} \cdot \frac{CF}{OF} \cdot \frac{OE}{BE} = 1. \qquad Th.\ 2.$$

$\triangle$ OCA is cut by the transversal DFY,

$$\therefore \frac{CY}{AY} \cdot \frac{AD}{OD} \cdot \frac{OF}{CF} = 1. \qquad Th.\ 2.$$

Multiplying these results and reducing, we have

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1;$$

$\therefore$ X, Y, and Z are collinear points in the sides of $\triangle$ABC. *Th.* 2.

Converse. If X, Y, and Z are collinear, to prove AD, BE, and CF concurrent. Let AD and CF meet in O, the theorem is established if B, E, and O are shown to be collinear.

Now $\triangle$s DAZ and FCX have their vertices on the concurrent straight lines DY, XY, and AY; therefore, as has just been shown, the points of intersection of their corresponding sides are collinear,

i.e., B, E, and O are collinear,

i.e., AD, BE, and CF are concurrent.

Theorem 4 is sometimes enunciated: *Co-polar triangles are co-axial; and conversely.*

[This theorem is due to Girard Desargues, an architect in Lyons (1593-1662). (See Poudra's *Œuvres de Desargues*, vol. i., pp. 413, 430.)]

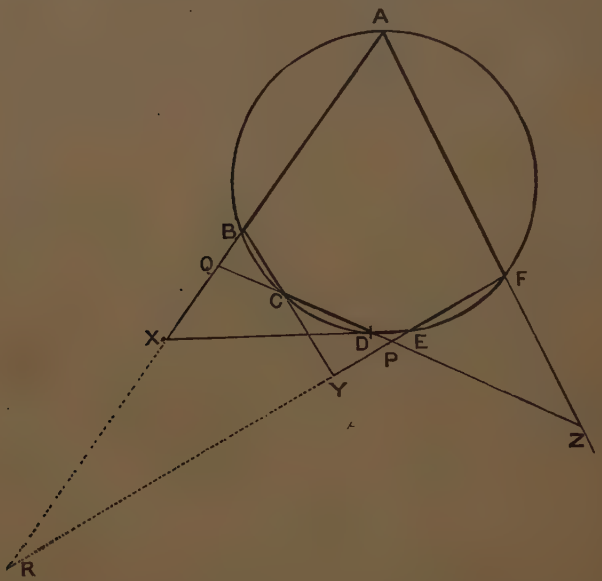
Ex. 30. Extend Theorem 4 to polygons whose vertices lie on concurrent straight lines.

Ex. 31. If the sides of a triangle turn round three collinear points while two of its vertices are constrained to remain on two given straight lines, find the locus of the remaining vertex.

Ex. 32. To inscribe in a given triangle, a triangle whose sides pass through three given collinear points.

THEOREM 5.

If a hexagon be inscribed in a circle, the points of intersection of the three pairs of opposite sides are collinear.



Let AB and DE meet in X, BC and EF in Y, and CD and FA in Z: then shall X, Y, and Z be collinear.

Produce, if necessary, three alternate sides AB, CD, and EF to form $\triangle$ PQR, and consider this triangle cut by the transversals BC, DE, and FA.

These give by Theorem 2 :

$$\frac{QB}{RB} \cdot \frac{RY}{PY} \cdot \frac{PC}{QC} = 1,$$

$$\frac{QX}{RX} \cdot \frac{RE}{PE} \cdot \frac{PD}{QD} = 1,$$

and $$\frac{QA}{RA} \cdot \frac{RF}{PF} \cdot \frac{PZ}{QZ} = 1.$$

Again, $PC \cdot PD = PE \cdot PF$, $QA \cdot QB = QC \cdot QD$, and $RE \cdot RF = RA \cdot RB$. *III.* 36, *Cor.*

Multiply together the first three equations, and reduce by means of the other three ;

$$\therefore \frac{QX}{RX} \cdot \frac{RY}{PY} \cdot \frac{PZ}{QZ} = 1,$$

$\therefore$ X, Y, and Z are collinear points in the sides of $\triangle PQR$. *Th.* 2.

The hexagon of this theorem is not necessarily convex : the points A, B, C, D, E, F may be joined in any order. There are 60 different figures which have these points as vertices. For we may always begin with A : beginning with any other point would give the same figures over again ; thus BCDEFA is the same figure as ABCDEF. We need therefore only consider the number of permutations of the remaining 5 letters : this we know from Algebra to be 120. But one half of these give the same figures as the other half ; thus ABCDEF and AFEDCB are the same figure. The number of really different hexagons is therefore 60. The student is recommended to draw a few of them.

Limiting Cases.—(i.) If one pair of the points coincide, the figure becomes a pentagon with a tangent to the circle at one of its vertices.

(ii.) If two pairs of the points coincide, the figure becomes a quadrilateral with tangents to the circle at two adjacent or at two opposite vertices.

(iii.) If three pairs of the points coincide, the figure becomes a triangle with tangents to the circle at its vertices.

In all these cases the theorem remains true if the suppressed sides be replaced by the corresponding tangents.

[Theorem 5 is enunciated in Pascal's *Essai pour les Coniques*, written in 1640, when its author was sixteen years of age. In a *Treatise on the Conic Sections* he deduced from this theorem more than 400 corollaries. After Pascal's death in 1662 this treatise was submitted to Leibnitz, who advised its publication. It was, however, never published, and we only know its contents from the analysis given in Leibnitz's letter. (See *Proceedings of the Edinburgh Mathematical Society*, vol. ii., p. 19.)]

Ex. 33. If A, B, C, D, E, F be any six points lying on two straight lines, and if AB and DE meet in X, BC and EF in Y, and CD and FA in Z, show that X, Y, and Z are collinear. (Pappus, vii. 139.)

Ex. 34. If any quadrilateral be divided into two quadrilaterals by a straight line, the points of intersection of the diagonals of the three quadrilaterals are collinear.

Ex. 35. Prove Ex. 9 by Th. 5.

HARMONIC RANGES AND PENCILS.

THE COMPLETE QUADRILATERAL AND QUADRANGLE.

Def. If C and D be two points in a range such that their position ratios with respect to other two points A and B are numerically equal but of opposite signs, ABCD is called a *harmonic range.*

$$i.e., \quad \frac{AC}{BC} = -\frac{AD}{BD}, \text{ or } \frac{AC}{BC} \div \frac{AD}{BD} = -1.$$

Def. C and D are called *harmonic conjugates* with respect to A and B.

Def. AB is said to be *divided harmonically* in C and D.

Def. AC, AB, and AD are said to be in *harmonic progression,* and AB is called the *harmonic mean* of AC and AD.

Cor. 1. If C bisects AB, D is at infinity. (*Prob.* 1, *Cor.* 2.)

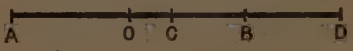
Cor. 2. If C coincides with B, so does D.

Cor. 3. If two harmonic ranges coincide in three corresponding points, they also coincide in the fourth.

In writing the range ABCD, it is not meant that the points are in that order in the diagram, but that C and D are harmonic conjugates with respect to A and B.

THEOREM 6.

If C *and* D *are harmonic conjugates with respect to* A *and* B, A *and* B *are harmonic conjugates with respect to* C *and* D.
Or, *If* ABCD *is a harmonic range, so is* CDAB.



Since C and D are harmonic conjugates with respect to A and B,

$$\frac{AC}{BC} = -\frac{AD}{BD}. \qquad \textit{Def.}$$

Interchange the means, and read all the segments in the opposite direction,

$$\therefore \quad \frac{CA}{DA} = -\frac{CB}{DB};$$

$\therefore$ A and B are harmonic conjugates with respect to C and D.

THEOREM 7.

If ABCD *be a harmonic range, and if* O *be the middle point of* AB, OB *is a geometric mean between* OC *and* OD.
Or, $OA^2 = OB^2 = OC \,.\, OD$. (Pappus, vii. 160.)

In the figure of Theorem 6, since ABCD is a harmonic range,

$$\frac{AC}{CB} = \frac{AD}{BD} \quad (\because \ CB = -BC),$$

$$\therefore \ \frac{AC - CB}{AC + CB} = \frac{AD - BD}{AD + BD}.$$

Now, since $AO = OB$,

$$AC - CB = 2 \cdot OC, \quad AD - BD = 2 \cdot OB,$$
$$AC + CB = 2 \cdot OB, \quad AD + BD = 2 \cdot OD.$$

Substitute these values and reduce,

$$\therefore \ OB^2 = OC \cdot OD.$$

Conversely: If $OB^2 = OC \cdot OD$, ABCD *is a harmonic range.*

This can be proved indirectly, or by reversing the preceding demonstration.

Ex. 36. Of the four points A, B, C, and D in the figure of Theorem 6, three always lie on one side of O.

Ex. 37. Show that the definition of harmonic progression given above agrees with that usually given in treatises on Algebra, viz.: "Three magnitudes are in harmonic progression if the first is to the third, as the difference between the first and second is to the difference between the second and third."

Ex. 38. If three straight lines be in harmonic progression, their reciprocals are in arithmetic progression.

Or, If ABCD be a harmonic range,

$$\frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB}.$$

Ex. 39. The projections of a harmonic range on any straight line form a harmonic range.

Ex. 40. The geometric mean of two straight lines is the geometric mean of their arithmetic and harmonic means.

Ex. 41. The centre of the circumscribed circle, the centroid, the centre of the nine point circle, and the orthocentre of a triangle, form a harmonic range.

(Steiner, *Gesammelte Werke*, i. 195.)

Ex. 42. If ABCD be any four collinear points, to find two points P and Q such that P and Q may be harmonic conjugates both with respect to A and B, and with respect to C and D.

Ex. 43. The straight lines which join the ends of a chord with any point in the circumference of a circle, divide harmonically the diameter perpendicular to the chord.

(Pappus, vii. 156.)

Ex. 44. If A and A_1 be harmonic conjugates with respect to B and B_1, and α and β the middle points of AA_1 and BB_1 respectively,

i. $AB \cdot A_1B_1 = AB_1 \cdot BA_1$.

ii. $AA_1 \cdot BB_1 = 2AB \cdot A_1B_1 = 2AB_1 \cdot BA_1$.

iii. $AA_1^2 = AB \cdot AB_1 - BA_1 \cdot A_1B_1$.

iv. $\dfrac{2}{AA_1} + \dfrac{2}{BB_1} = \dfrac{1}{BA_1} + \dfrac{1}{AB_1}$.

v. $A\alpha^2 + B\beta^2 = \alpha\beta^2$.

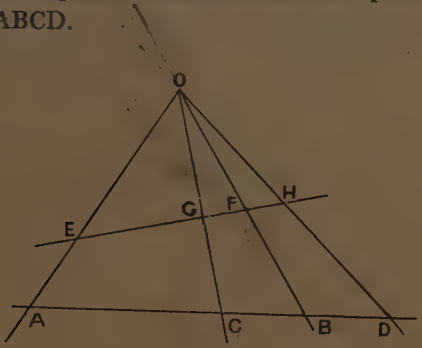
vi. $AB^2 = 2A\beta \cdot \alpha B$. (M. Dostor, *Nouv. Ann.*, vol. x.)

THEOREM 8.

If one transversal to a pencil of four rays be divided harmonically, so is every other transversal to that pencil.

(Particular case of Pappus, vii. 129.)

Def. Such a pencil is called a *harmonic pencil:* written thus, O . ABCD.



Given ABCD a harmonic range: it is required to prove EFGH also a harmonic range.

$$\frac{AC}{BC} = \frac{OA}{OB} \cdot \frac{\sin AOC}{\sin BOC'} \quad \frac{AD}{BD} = \frac{OA}{OB} \cdot \frac{\sin AOD}{\sin BOD'}$$

$$\frac{EG}{FG} = \frac{OE}{OF} \cdot \frac{\sin EOG}{\sin FOG'} \quad \frac{EH}{FH} = \frac{OE}{OF} \cdot \frac{\sin EOH}{\sin FOH}; \quad \textit{Th.}\ 1.$$

$$\therefore \frac{EG}{FG} \div \frac{EH}{FH} = \frac{\sin EOG}{\sin FOG} \div \frac{\sin EOH}{\sin FOH} = \frac{\sin AOC}{\sin BOC} \div \frac{\sin AOD}{\sin BOD}$$

$$= \frac{AC}{BC} \div \frac{AD}{BD} = -1;$$

$\therefore$ EFGH is a harmonic range. *Def.*

Cor. 1. Since the sine of an angle is equal to the sine of its supplement, the transversals may cut all or any of the rays on opposite sides of the vertex.

Cor. 2. If a transversal be drawn parallel to any one of the rays, the other three intercept equal segments on it.

(De La Hire, *Sectiones Conicæ*, p. 6.)

Let a ray parallel to OD meet the rays OA, OB, OC, and OD in P, Q, R, and ∞.

Then PQR∞ is a harmonic range;

$\therefore$ PR = RQ. *Prob.* 1, *Cor.* 2.

Cor. 3. If two pencils of four rays have equal angles at their vertices, and if one of them be harmonic, so is the other.

For they may be superposed.

Theorem 8 is usually enunciated as above, but it would be better to treat it as follows:—

Def. If OC and OD be two rays in a pencil such that their position ratios with respect to other two rays OA and OB are numerically equal but of opposite sign, O . ABCD is called a *harmonic pencil*,

$$i.e., \quad \frac{\sin AOC}{\sin BOC} = -\frac{\sin AOD}{\sin BOD},$$

$$\text{or,} \quad \frac{\sin AOC}{\sin BOC} \div \frac{\sin AOD}{\sin BOD} = -1.$$

Def. OC and OD are called *harmonic conjugates* with respect to OA and OB.

Cor. 1. If OC bisects the angle AOB, OD bisects its supplement.

Cor. 2. If OC coincides with OB, so does OD.

Cor. 3. If two harmonic pencils coincide in three corresponding rays, they also coincide in the fourth.

In writing the pencil O . ABCD, it is not meant that the rays occur in that order, but that OC and OD are harmonic conjugates with respect to OA and OB.

THEOREM 8 (Second form).

Every transversal to a harmonic pencil is divided harmonically.

See figure of Th. 8 above.

Let O . ABCD be a harmonic pencil, and AD any transversal; it is required to prove ABCD a harmonic range.

$$\frac{AC}{BC} = \frac{OA}{OB} \cdot \frac{\sin AOC}{\sin BOC}, \text{ and } \frac{AD}{DB} = \frac{OA}{OB} \cdot \frac{\sin AOD}{\sin BOD}; \quad Th.\ 1.$$

$$\therefore \frac{AC}{BC} \div \frac{AD}{BD} = \frac{\sin AOC}{\sin BOC} \div \frac{\sin AOD}{\sin BOD} = -1; \quad Def.$$

$\therefore$ ABCD is a harmonic range. *Def.*

Conversely: If ABCD *be a harmonic range, and* O *any external point,* O . ABCD *is a harmonic pencil.*

This can be proved by reversing the preceding demonstration.

THEOREM 9.

If two harmonic ranges be so placed that the straight lines joining three pairs of corresponding points are concurrent, the straight line joining the fourth pair passes through the same point. (De La Hire, pp. 7, 8.)

See figure of Th. 8.

Let AE, CG, and BF meet in O, and let OD cut EG in H_1.

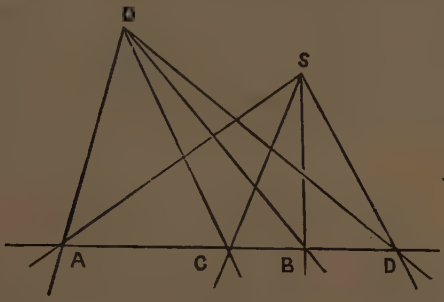
Then, since O . ABCD is a harmonic pencil, *Def.*
$EFGH_1$ is a harmonic range. *Th.* 8.
But EFGH is a harmonic range; *Hyp.*
$\therefore$ H and H_1 are the same point. *Prob.* 1, *Cor.* 2.
i.e., DH passes through O.

Cor. If one point of a harmonic range coincide with the corresponding point of another, the straight lines which join the other pairs of corresponding points are concurrent.

THEOREM 10.

If two harmonic pencils be so placed that the points of intersection of three pairs of corresponding rays are collinear,

the point of intersection of the fourth pair lies in the same straight line.



Let the pencils whose vertices are S and O have A, C, and B, the intersections of three pairs of corresponding rays collinear, and let the straight line ACB cut their fourth rays in D and D_1 respectively: the theorem is established if D and D_1 be shown to be the same point.

S . ABCD is a harmonic pencil; *Hyp.*

$\therefore$ ABCD is a harmonic range. *Th.* 8.

Similarly $ABCD_1$ is a harmonic range.

$\therefore$ D and D_1 are the same point.

Cor. If one ray of a harmonic pencil coincide with the corresponding ray of another, the points of intersection of the other pairs of corresponding rays are collinear.

Ex. 45. The sides of an angle and its internal and external bisectors form a harmonic pencil.

(De La Hire, p. 7.)

Ex. 46. If one ray of a harmonic pencil be the internal bisector, its conjugate is the external bisector of the angle between the other rays ; and conversely.

Ex. 47. If two conjugate rays of a harmonic pencil be at right angles to each other, they are the internal and external bisectors of the angle between the other rays.

(De La Hire, p. 7.)

Ex. 48. Show that Ex. 39 is a limiting case of Th. 8.

Ex. 49. The diagonals of a parallelogram, and straight lines through their point of intersection parallel to the sides of the parallelogram, form a harmonic pencil.

Ex. 50. If the bisector of the angle A of a triangle ABC meet the base in D, and if I and I_1 be the centres of the inscribed and escribed circles which lie in AD, $ADII_1$ is a harmonic range.

Ex. 51. If through a given point P there be drawn a transversal, meeting the sides of a given angle A in B and C, the locus of the harmonic conjugate of P with respect to B and C is a straight line through A.

Def. This locus is called the *polar* of P with respect to the angle A.

Ex. 52. The polars of a point with respect to the angles of a triangle meet the opposite sides in three collinear points.

p13

Def. A *complete quadrilateral* consists of four straight lines which intersect in six points (*vertices*).

The three straight lines which join opposite vertices are called *diagonals* (AC, BD, and EF, in the figure of Th. 11).

The diagonals determine a figure (GHK) called the *diagonal trilateral.*

Def. A *complete quadrangle* consists of four points which are joined by six straight lines (*sides*).

The three points in which opposite sides meet are called *diagonal points* (E, F, and G, in the figure of Th. 11).

The diagonal points determine a figure (EFG) called the *diagonal triangle.*

[The term *complete quadrilateral* was first used by Carnot, *Essai &c.*, p. 69. Steiner, *Systematische Entwickelung, &c.*, uses the terms *n*-side and *n*-point.]

An inspection of the figure of Th. 11 will show that the complete quadrilateral and the complete quadrangle differ only in their diagonal trilateral and triangle, and in the view we take of their formation. Thus we can inscribe a complete quadrangle in a circle, by taking four points on the circumference and joining them in every possible way; again, we can describe a complete quadrilateral about a circle, by drawing four tangents to the circle and producing them to meet in as many points as possible. But we cannot inscribe a complete quadrilateral in a circle, or

describe a complete quadrangle about one; for a circle cannot pass through the six vertices of the former, or touch the six sides of the latter.

To make the figure of Th. 11 better represent a complete quadrangle, draw the six lines AB, BC, CD, DA, AC, and BD thick, as these are the fundamental lines; draw EF, FG, and GE thin, these being the sides of the diagonal triangle; the other lines should be dotted, representing construction lines.

Ex. 53. The circles described about the triangles formed by the sides of a complete quadrilateral taken in threes are concurrent.

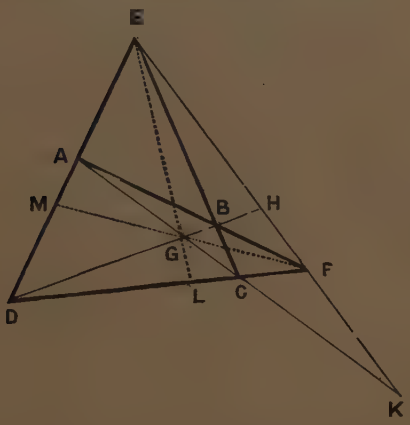
(W. Wallace, *Leybourn's Math. Repository*, N. S., vol. i., p. 22.)

Ex. 54. If a complete quadrangle be inscribed in a circle, the square on the line which joins the two diagonal points external to the circle is equal to the sum of the squares on the tangents to the circle from those points.

THEOREM 11.

In a complete quadrilateral the four points which lie in the same diagonal form a harmonic range.

In a complete quadrangle the four straight lines which pass through a diagonal point form a harmonic pencil.



Let ABCDEF be a complete quadrilateral; it is required to prove ACGK, DBGH, and EFHK harmonic ranges.

Join EG and produce it to meet CD in L.

Join FG and produce it to meet AD in M.

$\triangle$EDC is cut by the transversal ABF,

$$\therefore \frac{DF}{CF} \cdot \frac{CB}{EB} \cdot \frac{EA}{DA} = 1. \qquad Th.\ 2,$$

Also EL, DB, and CA are concurrent,

$$\therefore \quad \frac{DL}{CL} \cdot \frac{CB}{EB} \cdot \frac{EA}{DA} = -1. \qquad \textit{Th.}\ 3.$$

$$\therefore \quad \frac{DF}{CF} = -\frac{DL}{CL};$$

$\therefore$ DCLF is a harmonic range; *Def.*

$\therefore$ E . DCLF is a harmonic pencil; *Def.*

$\therefore$ ACGK and DBGH are harmonic ranges. *Th.* 8.

Again, consider the triangle EDF cut by the transversal ACK and the concurrent straight lines EC, DH, and FA, and we have, as above, EFHK a harmonic range.

If we look upon the figure as a complete quadrangle; ACGK, DBGH, and EFHK are harmonic ranges,

$\therefore$ E . ACGK, F . DBGH, and G . EFHK are harmonic pencils. *Def.*

Ex. 55. Apply Th. 11 to prove that the diagonals of a parallelogram bisect each other.

(See *Prob.* 1, *Cor.* 2, viii.)

Ex. 56. If AC in the figure of Th. 11 be bisected by BD, AC is parallel to EF.

Ex. 57. POQ is any angle; through any point A straight lines ABC, AB_1C_1, AB_2C_2, &c., are drawn, cutting OP in B, B_1, &c., and OQ in C, C_1, &c.: show that the points of intersection of the diagonals of the quadrilaterals BCC_1B_1, $B_1C_1C_2B_2$, &c., lie on a straight line passing through O.

PROBLEM 3.

Given three points in a harmonic range, to construct the fourth with the ruler only.

Given three rays of a harmonic pencil, to construct the fourth with the ruler only.

See the figure of Th. 11.

Let D, C, and F be the three points; or ED, EC, and EF the three rays.

In the former case take any external point E, and join ED, EC, and EF.

Then in both cases:

Through F draw any straight line cutting EC and ED in B and A respectively.

Join AC and BD meeting in G.

Join EG meeting CD in L.

L is the required point; or EL is the required ray.

∵ ABCDEF is a complete quadrilateral, DCLF is a harmonic range; *Th.* 11.

∴ E . DCLF is a harmonic pencil;

∴ L is the required point,

or EL is the required ray.

[This construction is given without proof in Frans Schooten's *Exercitationes Mathematicæ*, 1657, p. 174.]

Ex. 58. Given in the figure of Th. 11, D, L, and C, to construct F.

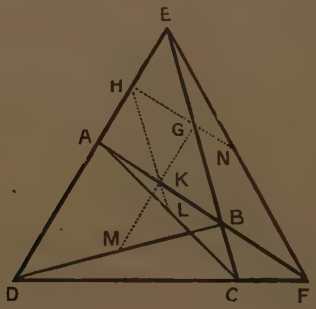
Ex. 59. Given D, L, and F, to construct C.

Ex. 60. Given L, C, and F, to construct D.

Ex. 61. To construct the polar of a given point with respect to a given angle.

THEOREM 12.

The middle points of the diagonals of a complete quadrilateral are collinear.



Let ABCDEF be a complete quadrilateral.

Bisect the sides of the △ EAB in G, H, and K.

Join GH, HK, and KG, and let these lines meet the diagonals in N, L, and M as in the figure.

We shall show that L, M, and N are the middle points of the diagonals, and that they are collinear.

∵ EA and EB are bisected,

GH is parallel to AB. *VI.* 2.

$\therefore$ N is the middle point of EF. *VI.* 2.

Similarly L and M are the middle points of AC and BD.

$\therefore$ HN, GN, KL, HL, GM, and KM are the halves of AF, BF, BC, EC, ED, and AD respectively.

Now $\triangle$ EAB is cut by the transversal DCF,

$$\therefore \frac{AF}{BF} \cdot \frac{BC}{EC} \cdot \frac{ED}{AD} = 1. \qquad \textit{Th.}\ 2.$$

Replace AF, BF, &c., by their halves HN, GN, &c.,

$$\therefore \frac{HN}{GN} \cdot \frac{KL}{HL} \cdot \frac{GM}{KM} = 1;$$

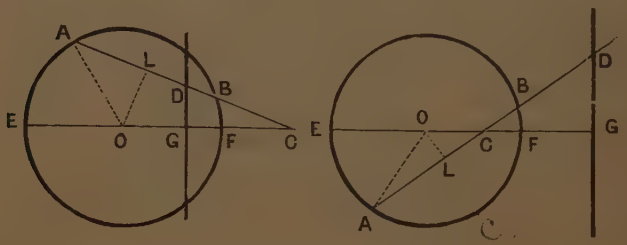
$\therefore$ L, M, N are collinear. *Th.* 2.

[Proposed by J. T. Connor, Lewes Academy, in *Ladies' Diary* for 1795-6. See Leybourn's Edition of the Mathematical Questions proposed in the *Ladies' Diary*, vol. iii., pp. 291-2.]

HARMONIC PROPERTIES OF THE CIRCLE.
POLE AND POLAR.

THEOREM 13.

If through any point C in the plane of a circle a transversal be drawn cutting the circle in A and B, the locus of D, the harmonic conjugate of C with respect to A and B, is a straight line.



Let EOCF be the diameter through C. Take G the harmonic conjugate of C with respect to E and F.

Join DG.

Then if DG be proved perpendicular to EF, the theorem will be established.

Draw OL perpendicular to AB, and join OA.

ABCD is a harmonic range, and AL = LB. *III.* 3.

$\therefore$ LC . LD = LA². *Th.* 7.

Similarly $OC \cdot OG = OF^2 = OA^2$.

Now $OC \cdot CG = \pm (OC \cdot OG - OC^2)$, *II.* 2 *or* 3.

$= \pm (OA^2 - OC^2)$,

$= \pm (AL^2 - LC^2)$, *I.* 47.

$= \pm (LC \cdot LD - LC^2)$,

$= LC \cdot CD$; *II.* 2 *or* 3.

$\therefore$ OLGD are concyclic; *III.* 35 *or* 36.

$\therefore \angle OGD$ is equal or supplementary to $\angle OLD$; *III.* 21 *or* 22.

$\therefore \angle OGD$ is a right angle.

Hence DG is the required locus.

Def. C is called the *pole* of DG, and DG the *polar* of C, with respect to the circle EAB.

[This theorem was known to Apollonius, who flourished between 250 and 200 B.C. It was used by De La Hire in his *Sectiones Conicæ* (1685). The word pole is an old name for a fixed point, *e.g.*, the North Pole; the term polar was introduced by Gergonne (*Annales*, i. 337).]

Cor. 1. Every transversal through the pole is divided harmonically by the circle, pole, and polar.

Cor. 2. The rectangle contained by the distances of the pole and polar from the centre is equal to the square on the radius of the circle.

Cor. 3. The polar of an internal point is wholly external, and that of an external point cuts the circle; and conversely.

Cor. 4. The polar of a point on the circumference is

the tangent at that point : and the pole of a tangent is its point of contact.

Cor. 5. The polar of the centre is the straight line at infinity.

Cor. 6. The pole of a diameter is a point at infinity.

Cor. 7. The angle between the straight lines joining any two points with the centre of a circle is equal to the angle between the polars of these points.

For the polars are perpendicular to the lines.

Ex. 62. Examine the cases in which the circle is replaced by a straight line (*i.e.*, becomes infinitely great).

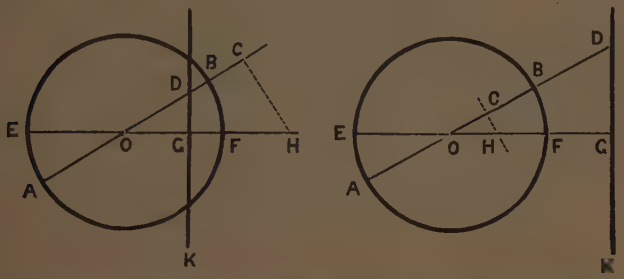
Ex. 63. If ABCD be a harmonic range, the ratio of the distances of any point on the circle whose diameter is CD from A and B is constant.

Ex. 64. To find the locus of points whose distances from two given points are in a given ratio. Examine the case in which the given ratio is unity.

THEOREM 14.

The polars of points which lie in a straight line pass through the pole of that line.

The poles of straight lines which pass through a point lie in the polar of that point.



Let K be any straight line, and H its pole with respect to a circle EBF.

In K take any point D.

Through D draw the diameter AB.

Draw the diameter EFG perpendicular to K.

Draw HC perpendicular to AB.

$\because$ $\angle$s HCD and HGD are right angles,

DCHG are concyclic; *III.* 22 *Converse.*

$\therefore$ $OC \cdot OD = OH \cdot OG$, *III.* 36, *Cor.*

$= r^2$, *Th.* 13, *Cor.* 2.

$= OB^2$;

$\therefore$ ABCD is a harmonic range; *Th.* 7.

$\therefore$ HC is the polar of D. *Th.* 13.

But D was any point in K;

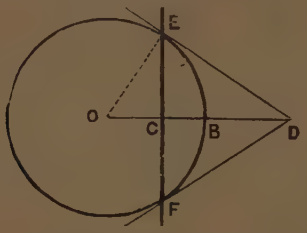
$\therefore$ the polar of every point which lies in K passes through H, the pole of K.

Similarly: The pole of every straight line which passes through H lies in K, the polar of H.

THEOREM 15.

The polar of an external point is the chord of contact of the tangents drawn to the circle from that point.

The pole of a chord of a circle is the point of intersection of the tangents at its ends.



Let D be an external point, and EF the chord of contact of the tangents drawn from D to the circle EBF.

It may be shown by Elementary Geometry that OD bisects EF at right angles.

Also $\angle$ OED = right angle; *III.* 18.

$\therefore$ OC . OD = OE^2, *VI.* 8.

$= r^2$;

$\therefore$ EF is the polar of D,

and D is the pole of EF. *Th.* 13, *Cor.* 2.

This theorem may also be proved thus:

Since a tangent is the polar of its point of contact, *Th.* 13, *Cor.* 4.

D lies on the polar of E.

Similarly, D lies on the polar of F.

$\therefore$ the polar of D passes through E and F, *Th.* 14.

i.e., EF is the polar of D;

$\therefore$ D is the pole of EF.

Cor. 1. If pairs of tangents be drawn to a circle from collinear points, their chords of contact are concurrent.

Cor. 2. If pairs of tangents to a circle be drawn from the ends of concurrent chords, their points of intersection are collinear.

Ex. 65. If from any point there be drawn to a circle a pair of tangents and a secant, the secant is divided harmonically by the circle and the chord of contact of the tangents. (Apollonius, *Conics*, iii. 37-40.)

Ex. 66. The polars of a harmonic range with respect to any circle form a harmonic pencil. (See *Th.* 13, *Cor.* 7.)

Ex. 67. The poles of a harmonic pencil with respect to any circle form a harmonic range. (See *Th.* 13, *Cor.* 7.)

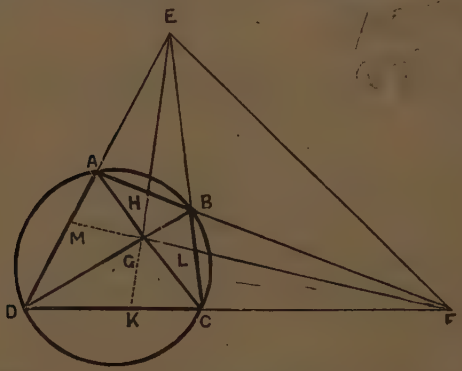
Ex. 68. Given a polygon and a circle, to construct another polygon whose sides shall be the polars of the vertices, and whose vertices shall be the poles of the sides, of the given polygon with respect to the given circle.

Examine the cases in which the given polygon is (i.) inscribed in the given circle, and (ii.) described about it.

THEOREM 16.

If a complete quadrangle be inscribed in a circle, each side of its diagonal triangle is the polar of the opposite vertex of that triangle.

Def. Such a triangle is called *self-conjugate.*



Let ABCD be a complete quadrangle inscribed in a circle, and EFG its diagonal triangle.

Let EG meet AB and CD in H and K, and FG meet BC and DA in L and M respectively.

E . DCKF is a harmonic pencil, *Th.* 11.

∴ F is the harmonic conjugate of H and K with respect to A and B, and C and D respectively;

∴ EG which passes through H and K is the polar of F. *Th.* 13.

Similarly, FG is the polar of E, and EF is the polar of G.

Cor. The centre of the circle is the orthocentre of EFG.

Ex. 69. An obtuse-angled triangle is self-conjugate with respect to the circle whose centre is the orthocentre of the triangle, and whose radius is a mean proportional between the distances of the orthocentre from any vertex of the triangle and from the side opposite that vertex.

Ex. 70. If a complete quadrilateral be described about a circle, its diagonal trilateral is self-conjugate.

Ex. 71. If a complete quadrangle have its vertices at the points of contact of the sides of a complete quadrilateral described about a circle, their diagonal triangle and trilateral are identical.

Ex. 72. If a complete quadrangle be inscribed in a circle, the straight line which joins the centre with one of its diagonal points is perpendicular to the straight line which joins the other two diagonal points.

PROBLEM 4.

To construct the polar of a point with the ruler only.

See the figure of Th. 16.

Let F be the given point.
Through F draw any two chords AB and CD.
Join AC and BD meeting in G.
Join AD and BC meeting in E.
Join EG.
EG is the polar of F. *Th.* 16.

Ex. 73. Draw the figure when F is within the circle.

PROBLEM 5.

To construct the pole of a given straight line with the ruler only.

Take any two points in the line and construct their polars as in Prob. 4.

The point in which these polars intersect is the required pole. *Th.* 14.

PROBLEM 6.

To draw a pair of tangents to a circle from an external point with the ruler only.

Construct the polar of the given point as in Prob. 4.

The part of this polar which is within the circle is the chord of contact of the required tangents. *Th.* 15.

Join the given point with the points in which the polar cuts the circle. These are the tangents required.

CROSS RATIOS OF RANGES AND PENCILS. ANHARMONIC RATIO.

Def. If there be four points in a range, the quotient of the position ratios of two of them with respect to the other two is called the *cross ratio* of the range.

Def. If there be four rays in a pencil, the quotient of the position ratios of two of them with respect to the other two is called the *cross ratio* of the pencil.

[Theorem 18 was known to the ancients; it is given in the *Mathematical Collection* of Pappus (about 300 A.D.). Chasles was the first to recognise its great importance, and named the quotients in the above definitions the *anharmonic ratio* (ἀνὰ ἁρμονική) of the range or pencil. The term *cross ratio* is due to Clifford.]

The preceding definitions are capable of twenty-four apparently different interpretations according to the pair of points or rays chosen for the base points or rays, and to the order in which the division of the position ratios is performed. In what follows, the expression (ABCD) is a contraction for "the cross ratio of the range ABCD," and (O . ABCD) for "the cross ratio of the pencil O . ABCD."

Again (ABCD) is always to mean $\frac{AC}{BC} \div \frac{AD}{BD}$, and (O . ABCD) to mean $\frac{\sin AOC}{\sin BOC} \div \frac{\sin AOD}{\sin BOD}$; or, in other words, the letters are always to be written so that the cross ratio is the quotient of the position ratio of the third point (or ray) by that of the fourth, with respect to the first and second.

The beginner is advised to practise writing down four letters at random, and then writing out the cross ratio in full, thus

$$(EFGH \text{ is}) \frac{EG}{FG} \div \frac{EH}{FH}.$$

The truth of the following statements can readily be proved by writing out the cross ratios and applying elementary algebra.

i. $(ABCD) = (BADC) = (CDAB) = (DCBA)$.
ii. $(ABDC) = (BACD) = (CDBA) = (DCAB)$.
iii. $(ACBD) = (BDAC) = (CADB) = (DBCA)$.
iv. $(ACDB) = (BDCA) = (CABD) = (DBAC)$.
v. $(ADBC) = (BCAD) = (CBDA) = (DACB)$.
vi. $(ADCB) = (BCDA) = (CBAD) = (DABC)$.

vii. The product of pairs taken one each from groups i. and ii., iii. and iv., or v. and vi., is equal to 1.

viii. The sum of pairs taken one each from groups i. and iii., ii. and v., or iv. and vi., is equal to 1.

Again, when A, B, and C are different points:

ix. If (ABCD) is positive, C and D are both within or both without the segment AB.

x. If (ABCD) is negative, C and D lie one within and one without the segment AB.

xi. If (ABCD) $= +1$, C and D coincide.

xii. If (ABCD) $= -1$, the range is harmonic.*

xiii. If (ABCD) $= 0$, B and D coincide.

xiv. If (ABCD) $= \infty$, A and D coincide.

Similar statements may be made about the pencil (O . ABCD).

Ex. 74. If (ABCD) $= k$, show that (ACBD) $= 1 - k$, (ADBC) $= \frac{k-1}{k}$, (ABDC) $= \frac{1}{k}$, (ACDB) $= \frac{1}{1-k}$, and (ADCB) $= \frac{k}{k-1}$.

Ex. 75. Deduce viii. above from the theorem: AB . CD + BC . AD + CA . BD $= 0$, where A, B, C, D are any four collinear points. (See Ex. 2.)

* Hence the theorems already given concerning harmonic ranges and pencils are particular cases of those which follow. The mode of proof adopted is such that the relation between the two groups can scarcely escape observation.

THEOREM 17.

(i.) *If two ranges with equal cross ratios have three corresponding points coincident, their fourth points are also coincident.*

Let (ABCD) = (ABCF),

i.e., $\frac{AC}{BC} \div \frac{AD}{BD} = \frac{AC}{BC} \div \frac{AF}{BF}$;

$\therefore \frac{AD}{BD} = \frac{AF}{BF}$;

$\therefore$ D and F coincide. *Prob.* 1, *Cor.*

(ii.) *If two pencils with equal cross ratios have three corresponding rays coincident, their fourth rays are also coincident.*

Let (O . ABCD = (O . ABCF),

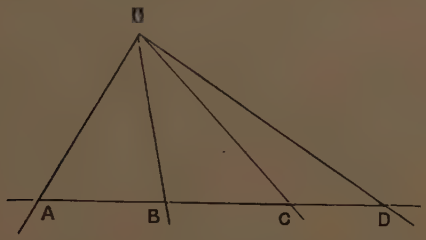
i.e., $\frac{\sin AOC}{\sin BOC} \div \frac{\sin AOD}{\sin BOD} = \frac{\sin AOC}{\sin BOC} \div \frac{\sin AOF}{\sin BOF}$;

$\therefore \frac{\sin AOD}{\sin BOD} = \frac{\sin AOF}{\sin BOF}$;

$\therefore$ OD and OF coincide. *Prob.* 2, *Cor.*

THEOREM 18.

If a pencil of four rays be formed by joining any external point with a range of four points, the cross ratio of the pencil is equal to that of the range.



Let ABCD be a range of four points and O any external point:

To prove (ABCD) = (O . ABCD).

$$\frac{AC}{BC} = \frac{OA}{OB} \cdot \frac{\sin AOC}{\sin BOC}, \text{ and } \frac{AD}{BD} = \frac{OA}{OB} \cdot \frac{\sin AOD}{\sin BOD}; \quad \textit{Th.}\ 1.$$

$$\therefore \frac{AC}{BC} \div \frac{AD}{BD} = \frac{\sin AOC}{\sin BOC} \div \frac{\sin AOD}{\sin BOD};$$

i.e., (ABCD) = (O . ABCD). *Def.*

Cor. 1. All transversals to a pencil of four rays have equal cross ratios. (Pappus, vii. 129.)

Cor. 2. If the rays of any number of pencils whose vertices are O, S, &c., pass through the points of a range ABCD, (O . ABCD) = (S . ABCD) = &c.

Cor. 3. If a transversal parallel to OD cut OA, OB, and OC respectively in P, Q, and R, $\frac{PR}{QR}$ is equal to the cross ratio of the range or pencil;

$$\text{For, } (PQR\ \infty) = (ABCD),$$

$$\textit{i.e.,}\ \frac{PR}{QR} = (ABCD) = (O\ .\ ABCD).$$

Cor. 4. Since the sine of an angle and the sine of its supplement are equal, all or any of the rays may be produced through the vertex, without altering the cross ratio of the pencil.

Cor. 5. If the angles between the rays of one pencil be equal or supplementary to those between the corresponding rays of another, the cross ratios of the pencils are equal.

Cor. 6. If four points be taken on the circumference of a circle, and pencils formed by joining them with a variable fifth point on the circumference, the cross ratios of these pencils are equal.

For the angles between the corresponding rays are equal or supplementary (III. 21 or 22).

THEOREM 19.

If two ranges of four points whose cross ratios are equal be so placed that the straight lines joining three pairs of corresponding points are concurrent, the straight line joining the fourth pair passes through the same point.

Cor. If two ranges of four points whose cross ratios are equal have one pair of corresponding points coincident, the straight lines which join the other three pairs of corresponding points are concurrent.

THEOREM 20.

If two pencils of four rays whose cross ratios are equal be so placed that the points of intersection of three pairs of corresponding rays are collinear, the point of intersection of the fourth pair lies in the same straight line.

Cor. If two pencils of four rays whose cross ratios are equal have one pair of corresponding rays coincident, the points of intersection of the other three pairs of corresponding rays are collinear.

For the proofs of Theorems 19 and 20 a very slight modification of those given for Theorems 9 and 10 will suffice.

As an example of the mode in which cross ratios may be used, we may add the following proof of Pascal's Theorem.

See the figure of Th. 5.

Let BC and ED meet in O.

Then (B . AEDC) = (F . AEDC). *Th.* 18, *Cor.* 6.

Now these pencils (whose rays are not all drawn in the figure) are cut by the transversals XEDO and ZPDC.

$\therefore$ (XEDO) = (ZPDC), because the cross ratios of the ranges are equal to those of the pencils. *Th.* 18.

Now these ranges have corresponding points coinciding in D;

$\therefore$ XZ, EP, and OC are concurrent.

i.e., X, Y, and Z are collinear. *Th.* 19, *Cor.*

Ex. 76. O is an angle and P is any point in its plane; through P transversals are drawn cutting the sides of O in AB, A_1B_1, &c.; through the point O any other straight line is drawn along which are taken $OC = OB$, $OC_1 = OB_1$, &c., show that AC, A_1C_1, A_2C_2, &c., are concurrent.

INVOLUTION.

Def. If A, A_1, B, B_1, C, C_1, and O be points in a range, such that $OA \cdot OA_1 = OB \cdot OB_1 = OC \cdot OC_1 = \pm k^2$, where k^2 is any constant, $AA_1BB_1CC_1$ are said to be a *system in involution.*

A and A_1, B and B_1, C and C_1 are called *conjugate points.*

O is the *centre,* and $\pm k^2$ the *constant of involution.*

When the constant is positive, two points D and D_1 may be found, such that each is its own conjugate: these points are called *double points,* and evidently lie one on each side of O at the distance k.

If the constant is negative, the double points are imaginary.

Ex. 77. The double points and any pair of conjugate points form a harmonic range.

THEOREM 21.

If $AA_1BB_1CC_1$ *be a system in involution, the cross ratio of any four of these points* (*not more than one pair being conjugates*) *is equal to that of their conjugates; and conversely.*

To prove, for instance, $(ABCC_1) = (A_1B_1C_1C)$.

$$\frac{AC}{BC} \div \frac{AC_1}{BC_1} = \frac{OC \pm OA}{OC \pm OB} \div \frac{OC_1 \pm OA}{OC_1 \pm OB} \quad \ldots \quad (1)$$

Now $OC = \pm k^2 \div OC_1$, $OA = \pm k^2 \div OA_1$, &c. *Def.*

Substitute these values in the right hand member of (1) which becomes on reducing

$$\frac{OC_1 \pm OA_1}{OC_1 \pm OB_1} \div \frac{OC \pm OA_1}{OC \pm OB_1} = \frac{A_1C_1}{B_1C_1} \div \frac{A_1C}{B_1C},$$

i.e., $(ABCC_1) = (A_1B_1C_1C)$.

Converse:

Let $(ABCC_1) = (A_1B_1C_1C)$, to prove $AA_1BB_1CC_1$ in involution.

Take any external point P, describe circles BPB_1, CPC_1 meeting again in Q; let PQ meet AA_1 in O.

Then rect. $OB \cdot OB_1 = OC \cdot OC_1$. *III.* 35.

Take A_2 so that $OA \cdot OA_2 = OB \cdot OB_1$.

Then $AA_2BB_1CC_1$ are in involution;

$\therefore$ $(A_2B_1C_1C) = (ABCC_1)$; *Th.* 21.

$= (A_1B_1C_1C)$; *Hyp.*

$\therefore$ A_2 coincides with A_1;

$\therefore$ $AA_1BB_1CC_1$ are in involution.

Def. If the points of a system in involution be joined with any external point, there is formed a *pencil in involution.*

Rays which pass through conjugate points are called *conjugate rays,* and those which pass through the double points are called *double rays.*

Ex. 78. The double rays of a pencil in involution and any pair of conjugate rays form a harmonic pencil.

THEOREM 22.

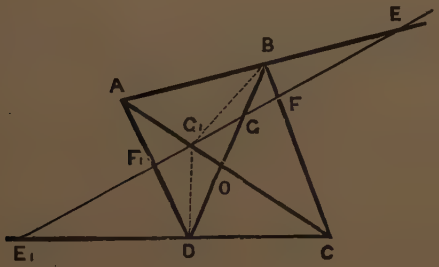
If $O.AA_1BB_1CC_1$ *be a pencil in involution, the cross ratio of any four rays (not more than one pair being conjugates) is equal to that of their conjugates; and conversely.*

This follows from Theorems 21 and 18.

Cor. If a pencil in involution be cut by any transversal, the points of section form a system in involution.

THEOREM 23.

Any transversal is cut in six points in involution by the sides of a complete quadrangle.



Join BG_1 and DG_1.

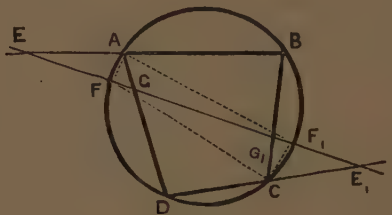
$(B.EFGG_1) = (B.ACOG_1) = (D.ACOG_1) = (D.F_1E_1GG_1)$
$= (D.E_1F_1G_1G)$; *Th.* 18.

$\therefore\ (EFGG_1) = (E_1F_1G_1G)$;

$\therefore$ $EE_1FF_1GG_1$ is a system in involution. *Th.* 21, *Conv.*

THEOREM 24.

Any transversal is cut in six points in involution by a circle and two pairs of opposite sides of an inscribed complete quadrangle.



Join AF, AF_1, CF, and CF_1.

$(A\,.\,EFF_1G)=(A\,.\,BFF_1D)=(C\,.\,BFF_1D)=(C\,.\,G_1FF_1E_1)$
$=(C\,.\,E_1F_1FG_1)$; *Th.* 18.

$\therefore\ (EFF_1G)=(E_1F_1FG_1)$;

$\therefore$ $EE_1FF_1GG_1$ is a system in involution.

Th. 21, *Conv.*

[The definition of involution and Th. 22, 23, and 24 were first given in Desargues' *Brouillon Project, &c.* (1639). See Poudra's *Œuvres de Desargues*, vol. i. pp. 119, 146, 171, and 186.]

Ex. 79. Prove the theorem, taking the sides AB, CD, AC, and BD.

RADICAL AXES. CO-AXAL CIRCLES.

Def. If through any point P in the plane of a circle there be drawn a chord AB, the rect. PA . PB is called the *potency* of P with respect to the circle.

Cor. We know from III. 35 and 36 that the rect. PA . PB is constant for all positions of the chord AB; that it is equal to $d^2 - r^2$ where d is the distance of P from the centre and r the radius of the circle; and that, when P is an external point, it is equal to the square on the tangent from P.

Also the *potency* is negative when P is within, and positive when P is without the circle.

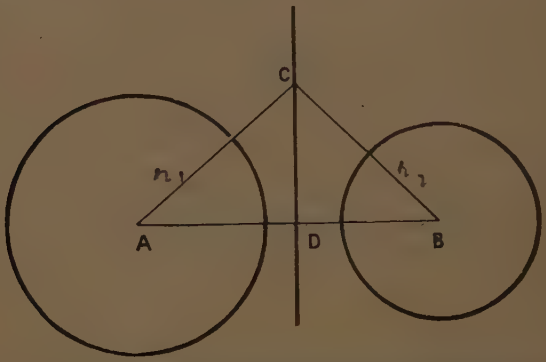
[The phrase "potency of a point with respect to a circle", is due to Steiner. See his *Gesammelte Werke*, i. p. 22.]

Ex. 80. Find the locus of a point whose potency with respect to a given circle is constant.

Ex. 81. Find the maximum and minimum values of the potency of a point with respect to a given circle.

THEOREM 25.

The locus of points of equal potency with respect to two circles is a straight line, perpendicular to the line joining their centres.



Let C be a point of equal potency with respect to the circles A and B, whose radii are r_1 and r_2 respectively.*

Draw CD perp. to AB.

Join CA and CB.

Since C is a point of equal potency with respect to the circles,

$$AC^2 - r_1^2 = BC^2 - r_2^2; \qquad \textit{Def.}$$

$$\therefore \quad AC^2 - BC^2 = r_1^2 - r_2^2.$$

$$\text{But} \quad AC^2 - BC^2 = AD^2 - BD^2. \qquad \textit{I.}\ 47.$$

$$\therefore \quad AD^2 - BD^2 = r_1^2 - r_2^2;$$

$$\therefore \quad \text{D is a fixed point.}$$

* In what follows the expression "the circle A" is frequently used for "the circle whose centre is A."

$\therefore$ the locus of C is a straight line perp. to AB, and dividing it into two segments such that the difference of their squares is equal to the difference of the squares on the radii.

Def. This straight line is called the *radical axis* of the circles.

[The term radical axis was introduced by L. Gaultier (*Journal de l'École Polytechnique*, 16 cahier, vol. ix.). He discovered this line by applying the principle of continuity to ascertain what becomes of the common chord of two intersecting circles, as the circles are gradually separated from each other until they have no longer any common point. The radical axis is sometimes called the common chord, real or imaginary, of the two circles according as they do or do not intersect.]

Cor. 1. If the circles intersect, their radical axis is the common chord.

Cor. 2. If the circles touch, their radical axis is the common tangent at their point of contact.

Cor. 3. If the circles neither cut nor touch, their radical axis is external to both.

Cor. 4. The radical axis of two circles may be found by the following construction.

Let A and B be the given circles.

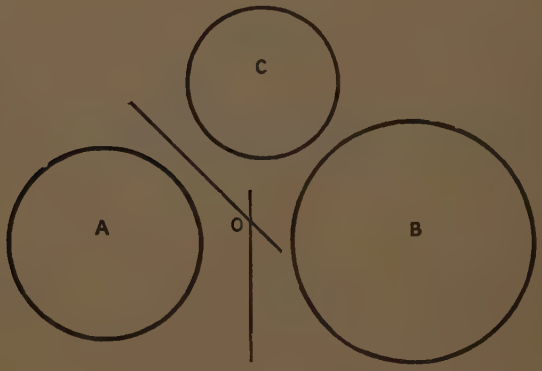
Draw any other circle to cut A and B.

Draw the common chords, and let them meet in P.

P is a point on the radical axis, which may then be found by drawing a perp. from P on the straight line joining the centres of A and B.

THEOREM 26.

The radical axes of three circles taken in pairs are concurrent.



Let A, B, and C be the three circles: and let the radical axis of A and B meet that of A and C in O.

Then, O is a point of equal potency with respect to A and B since it lies on their radical axis.

Similarly, O is a point of equal potency with respect to A and C.

$\therefore$ it is a point of equal potency with respect to B and C;

$\therefore$ the radical axis of B and C passes through O.

Def. O is called the *radical centre* of the three circles.

Cor. 1. If the centres of the circles are collinear, the radical centre is at infinity.

Cor. 2. Four or more circles have not, in general, a radical centre.

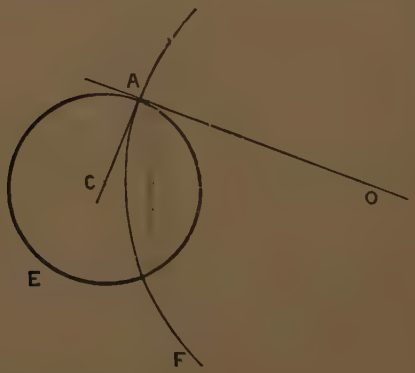
Def. Circles are said to cut each other *orthogonally* when the tangents at their points of section are perpendicular to each other.

THEOREM 27.

i. *If two circles cut each other orthogonally, a tangent to either at a point of section passes through the centre of the other.*

ii. *If two circles cut each other orthogonally, a radius of one drawn to a point of section is a tangent of the other.*

iii. *If with any point in a tangent to a circle as centre, a circle be drawn through the point of contact, the circles cut each other orthogonally.*



i. Let the circles E and F cut each other in A, and let AO and AC be the tangents at A.

Then, since the circles cut each other orthogonally, $\angle$ CAO is a right angle;

∴ AC passes through the centre of E, and AO passes through the centre of F. *III.* 19.

ii. Let CA be the radius of E drawn to A.

Draw AO tangent to E.

Then ∠ CAO is a right angle. *III.* 18.

i.e., AC is perp. to AO.

But the tangent to F at A is perp. to AO, since the circles cut orthogonally;

∴ AC is the tangent to F at A.

iii. Let AO be a tangent to the circle E, and F a circle whose centre is O (any point in the tangent), and which passes through A.

Then ∠ OAC is a right angle; *III.* 18.

∴ AC is a tangent to the circle F; *III.* 16.

∴ the circles cut each other orthogonally. *Def.*

Ex. 82. What does the radical axis become when the circles are replaced by straight lines or points (*i.e.*, when one or both of the circles become infinitely great or small)?

Ex. 83. What does the radical centre become when the circles are replaced by straight lines or points?

Ex. 84. The middle points of the four common tangents to two circles are collinear.

Ex. 85. If from any point on their radical axis a secant be drawn to each of two circles, the four points of section are concyclic.

Ex. 86. If from any point P there be drawn tangents to each of two circles, the difference of the squares on the tangents is equal to twice the rectangle contained by the distance between the centres of the circles and the distance of P from their radical axis.

Ex. 87. Find the locus of points such that the squares on tangents from them to each of two given circles may have a given difference.

Ex. 88. The radical axis of two circles is nearer the centre but further from the circumference of the less than of the greater circle.

Ex. 89. Two circles whose centres are A and B cut each other in C and D, a third circle is described passing through C and D and having its centre at the middle point of AB: show that the sum of the potencies of any point on the circumference of this circle with respect to the circles whose centres are A and B is zero.

Ex. 90. Find the locus of points whose potencies with respect to two given circles are equal numerically but of opposite sign. When is this possible?

Ex. 91. D, E, F are the feet of the perpendiculars from A, B, and C respectively, on the opposite sides of the triangle ABC: BC and EF meet in X, AC and DF in Y, and AB and DE in Z: show that XYZ is the radical axis of the circum- and nine-point- circles of ABC.

Ex. 92. What is the radical axis of two concentric circles?

Ex. 93. To describe a circle which bisects the circumferences of two given circles, and show that the locus of its centre is a straight line parallel to the radical axis of the circles.

Ex. 94. If three circles cut each other, their common chords are concurrent.

Ex. 95. If three circles touch each other, their common tangents at their points of mutual contact are concurrent.

Ex. 96. The orthocentre of a triangle is the radical centre of (i.) the circles whose diameters are the sides of the triangle, and (ii.) the circles whose diameters are the lines joining it with the angular points of the triangle.

Ex. 97. Each vertex of a triangle is the radical centre of (i.) three circles taken two from the first and one from the second group in the preceding exercise, and (ii.) three circles taken one from the first and two from the second of these groups.

Ex. 98. To draw a circle which bisects the circumferences of three given circles. (Use Ex. 93.)

Ex. 99. If two circles cut each other orthogonally, any diameter of one of them is divided harmonically by the other.

Ex. 100. If ABCD be a harmonic range, the circle whose diameter is AB cuts orthogonally every circle which passes through C and D.

Ex. 101. If two circles cut each other orthogonally, the polar of any point on the circumference of one of

them with respect to the other passes through the diametrically opposite point.

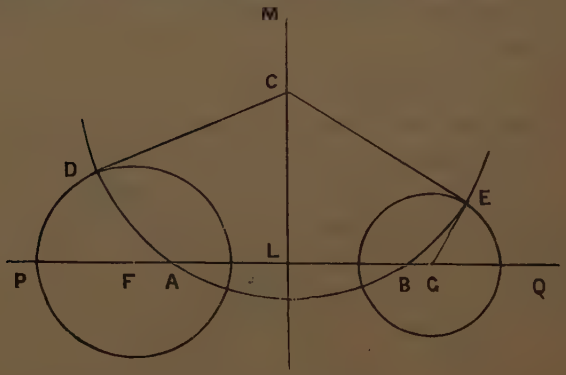
Ex. 102. If three circles touch each other, the circle through their points of contact cuts each of them orthogonally.

Ex. 103. To draw a circle cutting each of three given circles orthogonally.

THEOREM 28.

A straight line being given, there can be found systems of circles such that the given straight line is the radical axis of any pair.

Def. Such a system of circles is called *co-axal.*



i. Let PQ be the given straight line.

Draw any circle cutting PQ in A and B.

Through A and B draw any number of other circles.

Then PQ, being the common chord, is the radical axis of any pair. *Th.* 25, *Cor.* 1.

Def. Such a system we shall call an *intersecting system.*

ii. Let LM be the given straight line.

Take any external point F, and with F as centre describe a circle which does not cut LM.

Draw FL perp. to LM.

In LM take any point C.

From C draw CD a tangent to the circle F.

With centre C and radius CD describe a circle.

At any other point in this circle, such as E, draw a tangent meeting FL in G.

With centre G and radius GE describe a circle.

Join CE.

The circle G cuts the circle C orthogonally; *Th.* 27, iii.

∴ CE is a tangent to the circle G. *Th.* 27, ii.

Now CD = CE;

∴ C is a point of equal potency with respect to the circles F and G.

∴ C is on their radical axis. *Def.*

But CL is perp. to FG;

∴ CL is the radical axis of the circles F and G.

Similarly, by varying the position of E on the circumference of the circle C any number of circles can be found such that LM is the radical axis of any pair.

Def. Such a system we shall call a *non-intersecting system.*

Cor. 1. The circle C cuts the straight line FG in two points A and B, which may be looked upon as the smallest pair of circles belonging to the system.

Def. A and B are called the *limiting points* of the system.

Cor. 2. The circle C cuts orthogonally every circle belonging to the system.

Cor. 3. Every circle whose centre is in LM, and which passes through A and B, cuts orthogonally every circle of the system whose limiting points are A and B.

Cor. 4. There are two systems of co-axal circles for any given pair of points, one having these points for its limiting points, and the other such that every circle belonging to it passes through these points.

Cor. 5. Any circle belonging to one of these cuts orthogonally every circle belonging to the other.

Cor. 6. The locus of the centres of the circles of one of these systems is the radical axis of the other.

Ex. 104. In a non-intersecting system of co-axal circles the limiting points are harmonic conjugates with respect to the points in which any circle of the system cuts the straight line passing through them.

Ex. 105. The polar of one of the limiting points with respect to any circle of a co-axal system passes through the other limiting point.

Ex. 106. What represents the maximum pair of circles belonging to a system of co-axal circles?

Ex. 107. A common tangent to any pair of circles belonging to a non-intersecting system of co-axal circles subtends a right angle at either of the limiting points.

Ex. 108. (i.) The circles whose diameters are the three diagonals of a complete quadrilateral are co-axal; (ii.) the orthocentres of the four triangles made by taking the sides of the quadrilateral in threes lie on the radical axis of these circles; (iii.) hence show that the middle points of the diagonals of a complete quadrilateral are collinear.

SIMILITUDE.

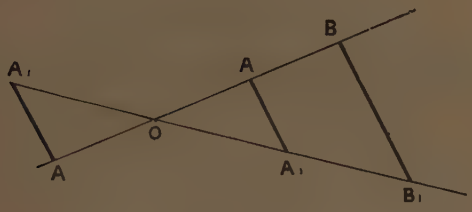
Def. If O be a fixed and A and B two variable points which move in such a manner that OAB is always a straight line and OA : OB constant, the loci of A and B are said to be *directly* or *inversely homothetic* according as A and B lie on the same or on opposite sides of O, *i.e.*, according as the ratio OA :.OB is positive or negative.

O is called the *direct* or *inverse centre of similitude* of the loci, and OA : OB their *ratio of similitude.*

Simultaneous positions of A and B are called *corresponding points.*

THEOREM 29.

The straight line which joins any two points in one of two homothetic figures is parallel to that which joins the corresponding points in the other, and has to it the ratio of similitude of the two figures.



Let O be the centre and $m : n$ the ratio of similitude, A and A_1 a pair of points in one figure, and B and B_1 the corresponding points in the other. It is required to show that

AA_1 is parallel to BB_1, and $AA_1 : BB_1 :: m : n$.

$OA : OB :: m : n :: OA_1 : OB_1$, *Hyp.*

$\angle AOA_1 = \angle BOB_1$;

$\therefore$ $\triangle$s AOA_1, BOB_1 are similar; *VI.* 6.

$\therefore$ $\angle OAA_1 = \angle OBB_1$;

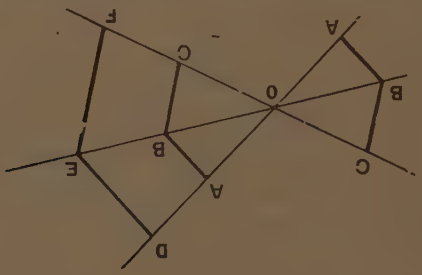
$\therefore$ AA_1 is parallel to BB_1. *I.* 27 or 28.

Also, $AA_1 : BB_1 :: OA : OB :: m : n$.

Cor. Parallel straight lines are both directly and inversely homothetic.

THEOREM 30.

If two polygons be homothetic, they are similar.



Let ABC . . . . , DEF . . . . be the two polygons, O their centre, and $m : n$ their ratio, of similitude.

AB is parallel to DE, and BC is parallel to EF, &c.

Th. 29.

$\therefore \quad \angle ABC = \angle DEF.$ *I.* 29.

Similarly, their other angles are equal.

Again, AB : DE : : $m : n$: : BC : EF. *Th.* 29.

Similarly, the sides about their other angles are proportional,

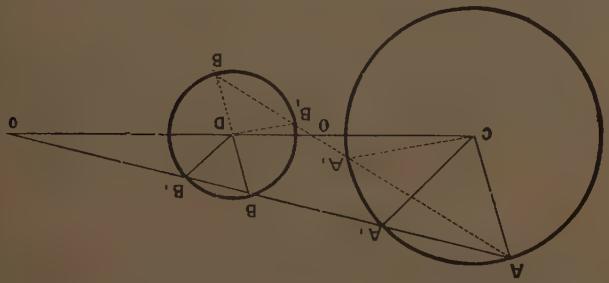
$\therefore$ the polygons are similar. *Def.*

Conversely : *If two similar polygons have their corresponding sides parallel they are homothetic* (*either directly or inversely*).

Cor. Similar regular polygons of an even number of sides are both directly and inversely homothetic, if their sides are parallel.

THEOREM 31.

Two circles are both directly and inversely homothetic: their centres of similitude divide the line joining their centres externally and internally in the ratio of their radii: and their ratio of similitude is the ratio of their radii.



Let C and D be the centres of the given circles, and r_1 and r_2 their radii.

Take O so that $OC : OD :: r_1 : r_2$.

Through O draw any straight line cutting the circles in A and B (A and B are either the points of section nearest O or those furthest from it), and also in A_1 and B_1.

Join CA, CA_1, DB, and DB_1.

Then $OC : OD :: r_1 : r_2 :: AC : BD$;

$$\angle AOC = \angle BOD.$$

Now $\triangle$s CAA_1 and DBB_1 are isosceles,

$\therefore$ $\angle$s OAC and OBD are both acute or both obtuse; *I.* 32.

$\therefore$ $\triangle$s OAC, OBD are similar; *VI.* 7.

$\therefore$ OA : OB : : $r_1 : r_2$;

$\therefore$ the loci of A and B, *i.e.*, the circles C and D, are homothetic, O is a centre of similitude, and $r_1 : r_2$ is the ratio of similitude. *Def.*

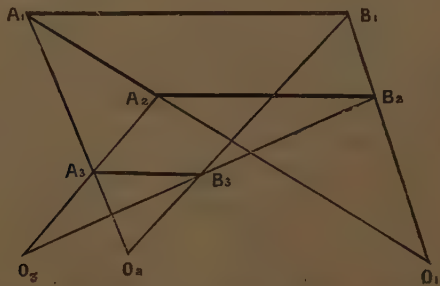
Cor. 1. The direct common tangents to a pair of circles pass through the direct, and the transverse common tangents through the inverse centre of similitude. (Limiting case when A and A_1 coincide.)

Cor. 2. The radii drawn to a pair of corresponding points are parallel.

Cor. 3. Tangents at a pair of corresponding points are parallel.

THEOREM 32.

If three figures be homothetic taken in pairs, their centres of similitude are collinear.



Let P_1, P_2, and P_3 be the three figures, and O_1, O_2, and O_3 the centres of similitude of P_1 and P_2, P_1 and P_3, and P_2 and P_3 respectively.

Also, let A_1B_1, A_2B_2, and A_3B_3 be pairs of corresponding points.

A_1B_1, A_2B_2, and A_3B_3 are parallel; *Th.* 29.

$\therefore$ $\triangle$s $A_1A_2A_3$ and $B_1B_2B_3$ have their vertices on three concurrent (in this case parallel) straight lines;

$\therefore$ the points of intersection of their corresponding sides (*i.e.*, O_1, O_2, and O_3) are collinear. *Th.* 4.

Def. $O_1O_2O_3$ is called an *axis of similitude*.

Cor. 1. If the figures be both directly and inversely homothetic, they have four axes of similitude.

Cor. 2. The six centres of similitude of three circles lie on four axes of similitude.

Otherwise: The six centres of similitude of three circles form the vertices of a complete quadrilateral which has the centres of the circles for the vertices of its diagonal triangle.

Ex. 109. To construct a polygon homothetic with a given polygon and having one of its sides of a given length.

Ex. 110. A_1, B_1, C_1 are the middle points of the sides *a*, *b*, and *c* respectively, of a triangle ABC: show that the triangles ABC and $A_1B_1C_1$ are homothetic with respect to G, the centroid of ABC.

Ex. 111. Hence show that the centres of the inscribed circles of ABC and $A_1B_1C_1$ are collinear with G, and the former twice as far from G as the latter.

Ex. 112. Prove that the same is true for the centres of the corresponding escribed circles; the centres of the circum-circles; and the orthocentres of ABC and $A_1B_1C_1$; and that the centroids of the two triangles coincide.

Ex. 113. The orthocentre of a triangle is the direct, and the centroid the inverse centre of similitude of its nine-point and circum-circles.

(Steiner, *Gesammelte Werke*, i. pp. 489-492.)

Ex. 114. The vertices of a triangle, and the points in which the internal and external bisectors of its angles meet the opposite sides, are the centres of similitude of its inscribed and escribed circles.

Ex. 115. The centres of two circles and their centres of similitude form a harmonic range.

Ex. 116. O and O_1 are two circles which do not touch, but may cut or not, and any point M on the circumference of O is joined with the centres of similitude by straight lines which meet the circumference of O_1 in four points: show that two of these points are at the ends of a diameter.

Ex. 117. If two circles touch, the corresponding ends of any pair of parallel diameters and the point of contact are collinear.

Ex. 118. If through a centre of similitude of two circles two secants be drawn cutting each circle in four points, the chords joining these points in corresponding

pairs are either parallel, or meet on the radical axis of the circles.

Ex. 119. If through a centre of similitude of two circles a secant be drawn cutting the circles in four points, and if through these points tangents be drawn to the circles, one of the diagonals of the quadrilateral so formed passes through the centre of similitude and the other coincides with the radical axis of the circles.

Ex. 120. The straight lines which join each centre of a system of three circles with the inverse centres of similitude of the other two are concurrent.

Ex. 121. If a variable circle touch two fixed circles, the points of contact and one of the centres of similitude of the fixed circles are collinear. (Poncelet, *Prop. Proj.*)

Ex. 122. Find the centres of similitude of two circles in the limiting cases in which the circles become

i. a point and a circle;
ii. two points;
iii. a straight line and a circle;
iv. two parallel straight lines;
and v. a straight line and a point.

Ex. 123. The radical axis of two circles is equally distant from the two polars of a centre of similitude with respect to the circles.

Ex. 124. The potencies of a centre of similitude of two circles with respect to the circles are proportional to the squares on the radii.

INVERSION.

Def. If O be a fixed and A and B two variable points which move in such a manner that OAB is always a straight line and rect. OA . OB constant, the loci of A and B are said to be *inverse* to each other.

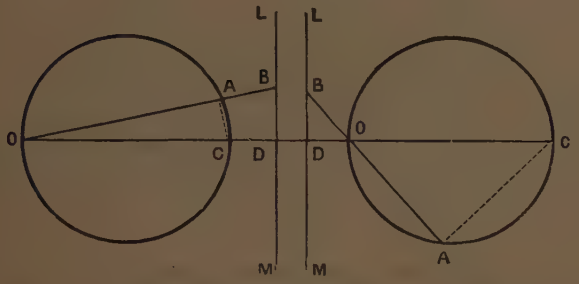
O is called the *centre of inversion*, and rect. OA . OB ($= \pm k^2$) the *constant of inversion*.

A and B are called *inverse points*, and lie on the same or opposite sides of O according as the constant of inversion is positive or negative.

[This method is the invention of Messrs Stubbs and Ingram (*Lond., Edin., and Dub. Phil. Mag.*, vol. xxiii. pp. 338-347). For an interesting comparison between similitude and inversion, see a paper by Dr Mackay in the *Proceedings of the Edinburgh Math. Society*, vol. vi. p. 69.]

THEOREM 33.

The inverse of a straight line is a circle passing through the centre of inversion.



Let LM be the line to be inverted, O the centre, and $\pm k^2$ the constant of inversion.

Draw OD perp. to LM.

In it take C the inverse of D

(*i.e.*, take C so that rect. OC . OD $= \pm k^2$).

In LM take any point B.

Join OB, and in it take A the inverse of B.

Join AC.

Then OA . OB $= \pm k^2 =$ OC . OD; *Const.*

$\therefore$ ABCD are concyclic; *III.* 35 or 36.

$\therefore$ $\angle$ OAC $=$ $\angle$ CDB $=$ a rt. angle; *III.* 21 or 22.

$\therefore$ the locus of A is the circle whose diameter is OC; *III.* 31.

i.e., the inverse of LM is a circle passing through O.

Conversely: *The inverse of a circle which passes through the centre of inversion is a straight line.*

Cor. The inverse of a polygon is a figure whose sides are arcs of circles all of which pass through the centre of inversion.

Ex. 125. Invert a circle into one of its own diameters (*i.e.*, choose O and k^2 so that this may be the case).

Ex. 126. Invert a circle into one of its tangents.

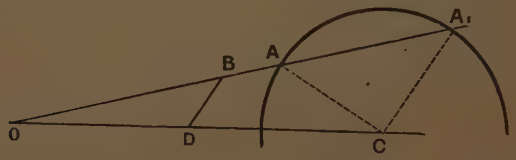
Ex. 127. For what values of k^2 will the inverse of a circle cut the circle, the centre of inversion being on the circumference?

Ex. 128. Draw the inverse of a pair of parallel straight lines.

Ex. 129. What is the inverse of a straight line through the centre of inversion?

THEOREM 34.

The inverse of a circle is, in general, a circle.



Let C be the circle to be inverted, O the centre, and $\pm k^2$ the constant of inversion.

Take any point A in the circumference of C.

Join OA, and in it take B the inverse of A.

Let A_1 be the other point in which OA cuts the circle C.

Join AC and A_1C.

Draw BD parallel to A_1C, meeting OC in D.

The locus of B is the inverse of the circle C.

For, rect. $OA \cdot OA_1 = \text{constant}$. *III.* 36.

rect. $OA \cdot OB = \text{constant}$. *Const.*

Hence by dividing, $OA_1 : OB = \text{constant}$.

Again, $\triangle$s OA_1C, OBD are similar; *VI.* 4.

$\therefore$ $OC : OD :: OA_1 : OB$;

$\therefore$ $OC : OD = \text{constant}$.

But O and C are fixed points;

$\therefore$ D is a fixed point.

Similarly, $A_1C : BD = \text{constant}$.

But A_1C is of constant length;

$\therefore$ BD is of constant length;

$\therefore$ the locus of B is a circle.

Cor. 1. If the circle C passes through O, OA_1 is zero, and therefore $OA_1 : OB = 0$.

$\therefore$ the radius DB becomes infinite, but the inverse of the point in which the circle C again cuts OC is finite, if $\pm k^2$ is finite. The inverse will in this case be a circle of infinite radius, *i.e.*, a straight line, as in Th. 33.

Cor. 2. The centre of inversion is a centre of similitude of the circle and its inverse.

Ex. 130. Draw the inverse of a circle, when the centre of inversion is within the circle.

Ex. 131. Draw the inverse of a circle, when its centre is the centre of inversion, and

i. $k^2 < r^2$;

ii. $k^2 = r^2$;

iii. $k^2 > r^2$.

Def. Straight lines are called *antiparallels* with respect to an angle, when they make equal angles with opposite sides of it.

Def. Straight lines are called *antiparallels* with respect to a straight line, when the angles they make with it on opposite sides of it are equal.

[The term antiparallel was first used by Antoine Arnauld in his *Nouveaux Éléments de Géométrie* (1667), p. 212.]

Ex. 132. Show that the second of the preceding definitions is a limiting case of the first.

Ex. 133. Straight lines which join points in the sides of an angle equidistant from the vertex are both parallel and antiparallel with respect to the angle.

Ex. 134. Perpendiculars to a straight line are both parallel and antiparallel with respect to the straight line.

Ex. 135. Two pairs of opposite sides of a complete quadrangle inscribed in a circle are antiparallels with respect to the angle between the third pair.

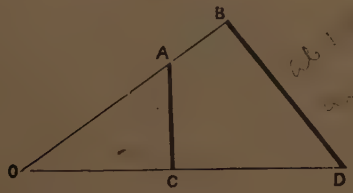
Ex. 136. The four points in which a pair of antiparallels with respect to an angle cut the sides of the angle are concyclic.

Ex. 137. DAE is a tangent to a circle at A; AB and AC are any two chords through A: show that DAE and BC are antiparallels with respect to the angle BAC.

Ex. 138. The straight lines which join the feet of the altitudes of a triangle are antiparallel to the sides of the triangle with respect to the angles of the triangle.

THEOREM 35.

Any two points and their inverses are concyclic: also the lines joining them are antiparallels with respect to the angle they subtend at the centre of inversion.



Let A and C be any two points, B and D their inverses, O the centre, and $\pm k^2$ the constant, of inversion.

$OA \cdot OB = \pm k^2 = OC \cdot OD$; *Hyp.*

$\therefore$ ABCD are concyclic. *III.* 35 or 36.

Also $\angle OAC = \angle ODB$; *III.* 21 or 22.

$\therefore$ AC and BD are antiparallels with respect to the angle AOC. *Def.*

THEOREM 36.

Tangents to a curve and its inverse at inverse points are antiparallels with respect to the straight line which contains the points of contact and the centre of inversion.

Let the points A and C in last fig. coincide, so will B and D, and the directions of AC and BD will be the directions of the tangents.

THEOREM 37.

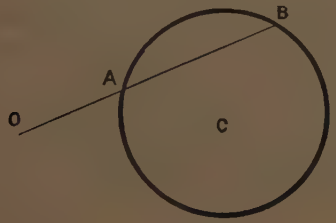
Two curves and their inverses cut each other at equal angles.

The angles between the curves are measured by the angles between their tangents at the point of intersection. These tangents are antiparallels, and therefore contain equal angles.

Cor. When two curves touch each other so do their inverses.

THEOREM 38.

Each circle of a group of circles is its own inverse, if the centre of inversion be a point of equal potency with respect to all the circles, and if the constant of inversion be equal to that potency.



Let C be one of the circles, and O a point whose potency with respect to each of the circles is $\pm k^2$.

Through O draw any straight line OAB, cutting the circle C in A and B.

The potency of O with respect to the circle C is OA . OB; *Def.*

$\therefore$ OA . OB $= \pm k^2$; *Hyp.*

$\therefore$ A and B are inverse points; *Def.*

$\therefore$ The locus of A is the inverse of the locus of B.

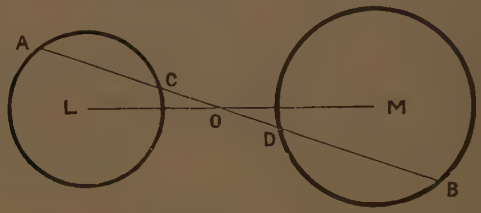
But the circle C is the locus of both A and B;

$\therefore$ the circle C is its own inverse.

Cor. In general not more than three circles can be inverted into themselves.

THEOREM 39.

Any two circles may be inverted into each other, by taking a centre of similitude as the centre of inversion, and a mean proportional between its potencies with respect to the two circles as the constant of inversion.



Let L and M be any two circles, O a centre of similitude, and $\pm p_1^2$ and $\pm p_2^2$ its potencies with respect to L and M respectively.

Through O draw any straight line cutting the circles in A, C and B, D respectively.

Because O is a centre of similitude,

$$OA : OB :: OC : OD;$$

$$\therefore \quad OA \cdot OD = OB \cdot OC.$$

Also, $OA \cdot OC = \pm p_1^2$ and $OB \cdot OD = \pm p_2^2$. *Hyp.*

Now $\pm k^2 = \pm\sqrt{p_1^2\, p_2^2}$, *Hyp.*

$$= \pm\sqrt{OA \cdot OC \cdot OB \cdot OD},$$

$$= \pm\sqrt{OA^2 \cdot OD^2}, \ (\because\ OA \cdot OD = OB \cdot OC),$$

$$= \pm OA \cdot OD;$$

$\therefore$ A and D are inverse points; *Def.*

$\therefore$ the circles L and M are each the inverse of the other.

Ex. 139. The centres of inverse circles are not inverse points.

Ex. 140. A non-intersecting system of co-axal circles inverts into a system of concentric circles, when one of the limiting points is taken as the centre of inversion.

Ex. 141. What is the inverse of the co-axal system, the circles of which pass through the limiting points of the system in the preceding exercise, for the same centre and constant of inversion?

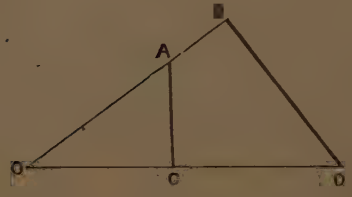
Ex. 142. A, B, and C are three fixed points in a straight line, a circle of variable radius touches this straight line at C, and from A the other tangent is drawn to this circle meeting it at D, BD is joined and meets the circle at P; show that the locus of P is the inverse of the circle whose centre is A and radius AC, B being the centre and BC^2 the constant of inversion.

Ex. 143. Every straight line through the centre of inversion which touches a curve touches also its inverse.

THEOREM 40.

A *and* C *being any two points and* B *and* D *their inverses,* O *the centre and* $\pm k^2$ *the constant of inversion :*

$$BD = \frac{\pm k^2}{OA \cdot OC} \cdot AC.$$



BD and AC are antiparallels ; *Th.* 35.

$\therefore$ $\triangle ODB$ similar to OAC ; *VI.* 4.

$$\therefore \frac{BD}{AC} = \frac{OD}{OA} = \frac{OC \cdot OD}{OA \cdot OC} = \frac{\pm k^2}{OA \cdot OC};$$

$$\therefore BD = \frac{\pm k^2}{OA \cdot OC} \cdot AC.$$

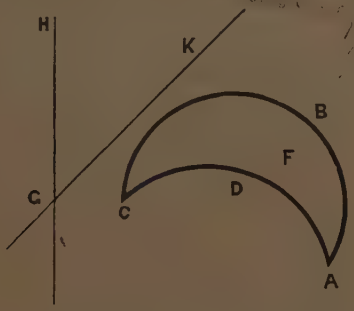
[This theorem is one of great importance, as it enables us to deduce the metrical properties of a figure from those of its inverse.

Metrical properties are those which involve the lengths of lines as contrasted with *descriptive* properties which involve only the relative position of points and lines.

The Ancient Geometry more specially deals with the former, and Modern Geometry with the latter. Of the two following theorems, the first is descriptive and the second metrical.]

THEOREM 41.

The locus of the point of contact of a pair of circles which touch each other and each of two intersecting circles, is a circle.



Let F be the point of contact of circles L and M (not shown in the figure) which touch each other and each of the intersecting circles ABC and ADC.

It is required to find the locus of F.

Take A as the centre of inversion, and any value for k^2.

The circles ABC and ADC invert into the intersecting straight lines GK and GH. *Th.* 33 *and* 37.

Now the circles L and M invert into two others, P and Q, which touch GH and GK, and also each other in a point R, the inverse of F (these are not shown in the figure). *Th.* 34 *and* 37, *Cor.*

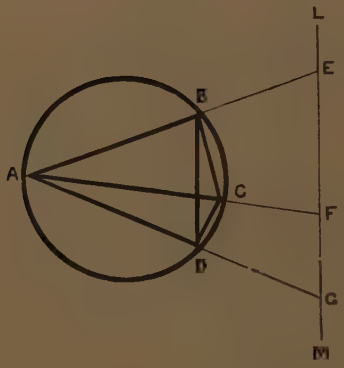
But the locus of R is, by elementary geometry, one of the bisectors of the angle G;

$\therefore$ the locus of F is the inverse of that line. Hence that locus is a circle passing through A, and making equal angles with the circles ABC and ADC.

Th. 33 *and* 37.

THEOREM 42.

To prove Ptolemy's Theorem (*VI. D.*) *by means of its inverse.*



Take A, one of the vertices of the quadrilateral, as the centre of inversion and any value for k^2.

Let the circle invert into LM. *Th.* 33.

Produce AB, AC, and AD to meet LM in E, F, and G, the inverses of B, C, and D respectively.

Now $$EF = \frac{k^2}{AB \cdot AC} \cdot BC;$$

$$FG = \frac{k^2}{AC \cdot AD} \cdot CD;$$

$$\text{and } EG = \frac{k^2}{AB \,.\, AD} \,.\, BD. \qquad Th.\ 40.$$

But $EG = EF + FG$.

$$\therefore \frac{k^2}{AB \,.\, AD} \,.\, BD = \frac{k^2}{AB \,.\, AC} \,.\, BC + \frac{k^2}{AC \,.\, AD} \,.\, CD;$$

which readily reduces to

$$AC \,.\, BD = AD \,.\, BC + AB \,.\, CD.$$

The proposed theorem is thus made to depend on the extremely simple, indeed axiomatic, proposition $EG = EF + FG$.

Ex. 144. The inverse of a sphere with respect to a point in its surface is a plane.

Ex. 145. If three circles intersect in a point, the sum of the angles at the other three points of section is four right angles. (The angles are made by tangents drawn in the same cyclical direction.)

Ex. 146. By taking a point on the circumference other than one of the vertices of the quadrilateral as the centre of inversion, show that Ptolemy's Theorem is the inverse of this theorem, "If ABCD be a range of four points, $AB \,.\, CD + AD \,.\, BC = AC \,.\, BD$."

Note.—This latter theorem is really a limiting case of Ptolemy's Theorem, in which the circle has become infinite while the vertices remain finite.

Ex. 147. ABC is an isosceles triangle, from the vertex A a straight line is drawn to meet the base in D and the circum-circle in E; show that $AB^2 = AD \,.\, AE$.

(Invert the circle into BC.)

METHOD OF RECIPROCAL POLARS.

Another method of transformation of figures is that known as the method of *reciprocal polars*. It is founded on the dual relation between points and straight lines proved in Th. 14, and was suggested to Poncelet by the relation between Ths. 5 and 43.

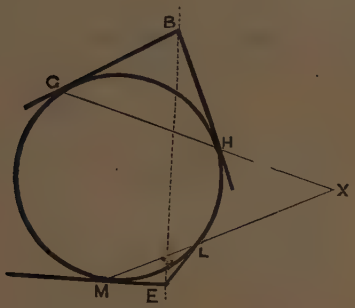
In the plane of a given rectilineal figure describe any circle; then describe another rectilineal figure such that each of its sides is the polar of a vertex of the first figure with respect to the circle.

It follows from Th. 14 that each vertex of the second figure is the pole of a side of the first. Such figures are called *polar reciprocals*.

In the limit when the figures become curves, they are so related that the tangent at any point in one of them is the polar of the corresponding point in the other.

THEOREM 43.

If a hexagon be described about a circle, the straight lines which join the three pairs of opposite vertices are concurrent.



Let ABCDEF be a hexagon described about a circle; it is required to prove AD, BE, and CF concurrent.

[Only a portion of the figure is drawn.]

Join GHKLMN, the points of contact.

Produce GH and ML to meet in X, HK and MN to meet in Y, and KL and NG to meet in Z.

Now, since GHKLMN is an inscribed hexagon, XYZ are collinear. *Th.* 5.

Again, B and E are the poles of GH and LM; *Th.* 15.

∴ BE is the polar of X. *Th.* 14.

Similarly, AD and CF are the polars of Z and Y.

But the polars of collinear points are concurrent; *Th.* 14.

∴ AD, BE, and CF are concurrent.

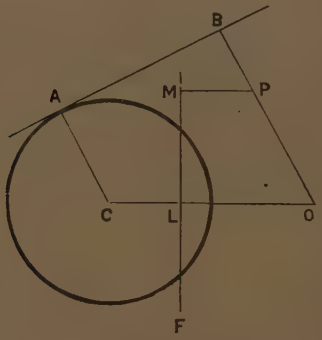
[This theorem is due to Brianchon, *Journal de l'École Polytechnique*, vol. vi., 1806. It is the polar reciprocal of Th. 5.]

Ex. 148. Deduce from Th. 43, by supposing one or more of the angles of the hexagon to become = 180°, a theorem

about a circumscribed pentagon, two theorems about a circumscribed quadrilateral, and a theorem about a circumscribed triangle. (See p. 22.)

THEOREM 44.

The polar reciprocal of a circle with respect to another circle is a conic section.



Let C be the circle whose polar reciprocal with respect to the circle O (not drawn in the figure) is to be found.

Draw LF the polar of the point C with respect to the circle O.

At any point A draw a tangent to the circle C.

Draw OB perpendicular to the tangent, and in it take P the pole of AB.

The locus of P is the required curve.

Join OC and CA, and draw PM perpendicular to FL.

$OP \cdot OB = r^2 = OL \cdot OC$; *Th.* 13, *Cor.*

$\therefore$ $OP : OL :: OC : OB$.

Also, the quadrilateral OPML is equiangular to OCAB, ($\because$ they have an angle common and other two angles each right angles);

$\therefore$ they are similar; *VI.* 4 *and* 6.

$\therefore$ $OP : PM :: OC : CA$.

But $OC : CA$ is constant.

$\therefore$ $OP : PM$ is constant;

$\therefore$ the locus of P is a conic section, whose focus is O and directrix FL.

Ex. 149. When will the conic be a parabola : an ellipse : a hyperbola ?

Ex. 150. The ratio of the distances of any two points from the centre of a circle is equal to the ratio of the distances of each from the polar of the other.

[Salmon.]

LINKAGES.

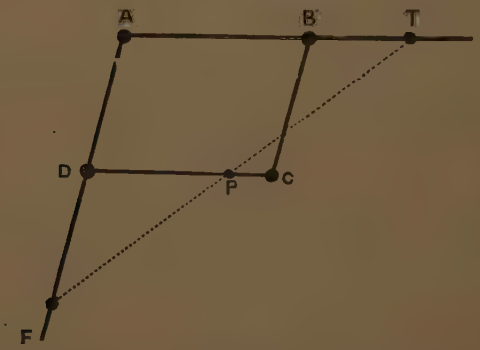
A jointed framework of bars or rods is called a *linkage*; each rod is called a *link*.

The simplest linkage consists of a single link. If one end of the link be fixed, the other end describes a circle.

By means of more complicated linkages other curves may be described.

Two linkages are of interest to us from their intimate connection with the sections on Similitude and Inversion.

I. *The Pantagraph.*



Let ABCD be a parallelogram having the sides AB and AD produced, composed of rods with compass joints

at A, B, C, and D. Means are provided (by graduating the arms) for fixing the points F, P, and T in such a manner that FA : FD = AT : DP = any given ratio. The whole apparatus is then fixed to a firm base at F, a pencil is fixed at P, and a tracing point at T.

Since the triangles FAT and FDP have an angle of each equal, and the sides about these angles proportional, the triangles are similar (VI. 6). Hence F, P, T are collinear, and FT : FP in the given ratio, for any displacement of the linkage about F.

P will thus describe a figure homothetic with that traced out by T, but less in the given ratio.

To copy a drawing on a reduced scale, go over its outline with T, and P will describe a similar outline on a paper placed under it.

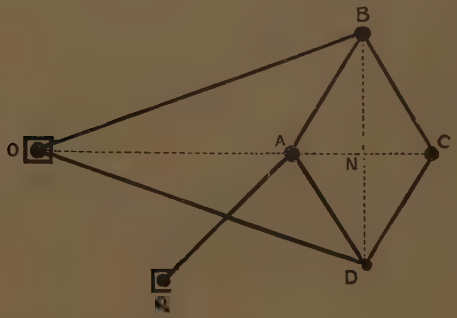
To enlarge the scale it is sufficient to interchange P and T.

[The inventor of the pantagraph seems to have been Christopher Scheiner; see his *Pantographice, seu Ars Delineandi Res Quaslibet per Parallelogrammum Lineare seu Cavum Mechanicum Mobile*, Rome, 1631.]

II. *To draw a straight line by means of a linkage.*

In the first place, observe the difference between the usual methods of drawing a straight line and a circle. In the case of the circle, we carry out the geometrical definition by means of compasses; and if we use, as is sometimes done, a rod carrying a sharp point and a

pencil, we are using a linkage of one link. In drawing a straight line by means of a ruler, we only copy a straight line previously made for us, just as if we were to use a penny to draw a circle. To discover a linkage which would actually describe, and not merely copy, a straight line, was a problem which mechanicians long endeavoured to solve. James Watt gave an approximate solution of it in his famous "parallel motion," an arrangement which is still in use in most stationary engines. It was not till 1864 that Peaucellier, a French officer of engineers, solved the problem by the following beautiful application of Th. 34 :



ABCD is a linkage of four equal links; to the opposite corners, B and D, are joined two equal links, OB and OD (either longer or shorter than the sides of the rhombus). These six links form what is called a *Peaucellier cell*, or an *inverter*. If O be fixed, A and C are inverse

points with respect to O. For O, A, and C are from the symmetry of the figure collinear.

Draw BN perpendicular to OC.

BN bisects AC; *I.* 26.

$\therefore$ $OA \cdot OC = ON^2 - NC^2$, *II.* 6.

$= OB^2 - BC^2$, *I.* 47.

= a constant.

$\therefore$ A and C are inverse points.

To cause the point C to describe a straight line, it is only necessary to cause A to describe a circle which passes through O. This is readily done by adding a seventh link of any convenient length, and fixing it at a point Q, such that QO is equal to QA. [See Th. 33 converse, or Th. 34, Cor. 1.]

For a very simple and interesting description of these and other linkages the reader is referred to Kempe's *How to draw a Straight Line* (Nature Series, Macmillan & Co.), where will be found an account of a five-link linkage discovered by Prof. Hart of Woolwich, by which a straight line can be drawn. [See Ex. 199 and 200.]

MISCELLANEOUS EXERCISES

Ex. 151. A, B, C, D, and E are any five points in a range; show that $EA^2 . BC . CD . DB + EC^2 . DA . AB . BD = EB^2 . CD . DA . AC + ED^2 . AB . BC . CA$.

(Chasles, *Aperçu &c.*, p. 176.)

Ex. 152. Deduce Ex. 3 ii. from Ex. 151.

(Consider D at infinity.)

Ex. 153. Deduce Ex. 2 from Ex. 151.

(Consider E and A co-incident.)

[Ex. 2 is a particular case of a theorem by Matthew Stewart, 1746.]

Ex. 154. In the figure of Theorem 6, show that BC, BA, BD; CA, CD, CB; and DA, DC, DB are in harmonical progression.

Ex. 155. Deduce Ex. 22 and 23 from Ex. 12.

Ex. 156. If in the triangle ABC, the altitude, the angle bisector, and the median, through A, B, and C respectively, are concurrent; show that

$$\frac{b^2 - c^2}{a^2} = \frac{a - c}{a + c}.$$

Ex. 157. In any triangle, the straight lines which join

a vertex with the middle points of the medians through the other vertices trisect the opposite side. .

Ex. 158. ABC is a triangle inscribed in a circle; its sides produced are cut by a transversal in D, E, and F, and from these points tangents are drawn to the circle; show that the product of the three tangents is equal to BD . CE . AF or CD . AE . BF.

Ex. 159. E, F, G, H are four points in the sides of a quadrilateral ABCD such that EF, GH, and AC are concurrent; show that EH, FG, and BD are also concurrent.

Ex. 160. ABC is a triangle whose sides are cut by a transversal in D, E, and F. If D_1, E_1, and F_1 be harmonic conjugates of D, E, and F with respect to the vertices of the triangle, show that

(i.) AD_1, BE_1, and CF_1 are concurrent;

(ii.) AD_1, BE, and CF are concurrent;

(iii.) AD, BE_1, and CF are concurrent;

and (iv.) AD, BE, and CF_1 are concurrent.

Ex. 161. With the notation of Ex. 160, show that

(i.) D_1, E_1, and F are collinear; (ii.) D_1 E, and F_1 are collinear; and (iii.) D, E_1, and F_1 are collinear.

Ex. 162. Deduce from Ex. 160 that the medians of a triangle are concurrent.

Ex. 163. AB and CD are straight lines of given length, it is required to place them so that ABCD may be a harmonic range.

Ex. 164. If (ABCD) = (O . ABCD) = n, show that

$$\frac{n-1}{AB} = \frac{n}{AC} - \frac{1}{AD};$$

and that

$$\frac{n-1}{\tan AOB} = \frac{n}{\tan AOC} - \frac{1}{\tan AOD}.$$

(Attributed to Möbius by Baltzer.)

Ex. 165. If ABCD and O . ABCD be a harmonic range and pencil, show that

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD};$$

and that

$$\frac{2}{\tan AOB} = \frac{1}{\tan AOC} + \frac{1}{\tan AOD}.$$

Ex. 166. A variable tangent to a circle is cut by four fixed tangents; show that the cross ratio of the points of intersection is constant.

Ex. 167. a_1, a_2 . . . a_n are points in one range, and b_1, b_2 . . . b_n points in another; what is the condition that these ranges can be so placed that a_1b_1, a_2b_2 . . . a_nb_n may be concurrent?

Ex. 168. If three right angles have a common vertex, their sides form a pencil in involution.

Ex. 169. G is the middle point of a chord which meets one pair of opposite sides of an inscribed quadrilateral in C and D and the other pair in E and F; show that if GC = GD, GE = GF.

Ex. 170. P is any point in the altitude AD of a triangle ABC; through it are drawn BPE and CPF meeting the opposite sides in E and F; show that DE and DF are equally inclined to AD.

Ex. 171. If two tangents be drawn to a circle, any other tangent is divided harmonically by its point of contact, the former tangents, and their chord of contact.

Ex. 172. Four points are taken on the circumference of a circle; the straight lines which join them in every possible way intersect in other three points; show that the straight line which joins any one of these three points with the centre is perpendicular to the straight line which joins the other two.

Ex. 173. The angle at the centre of a circle subtended by any two points is equal to the angle between their polars.

Ex. 174. The cross ratio of four collinear points is equal to the cross ratio of their polars.

Ex. 175. No acute-angled triangle can be self-conjugate.

Ex. 176. To construct the circle with respect to which a given triangle is self-conjugate.

Ex. 177. To construct a circle, having given a tangent, any other straight line in the plane and its pole with respect to the circle.

Ex. 178. To construct a circle, having given a point on its circumference, any other point in the plane, and its polar with respect to the circle.

Ex. 179. The polars of any point in the radical axis of two circles with respect to these circles intersect on the radical axis.

Ex. 180. To describe a circle which shall pass through a given point and cut orthogonally two given circles.

Ex. 181. The circles described on the diagonals of a complete quadrilateral as diameters cut orthogonally the circle about the diagonal triangle.

Ex. 182. If two circles have four common tangents, the circles described on these tangents as diameters are co-axal.

Ex. 183. If three circles have a common radical axis, one of them divides harmonically the common tangents of the other two.

Ex. 184. To find a point and a straight line which are pole and polar with respect to each of two given circles.

Ex. 185. If a given point and straight line are pole and polar with respect to each of two given circles, there is an infinite series of circles with respect to all of which they are pole and polar.

(Poncelet, *Prop. Proj.*, vol. i., p. 43.)

Ex. 186. P and Q are the limiting points, and Z the point at infinity in the radical axis in a system of co-axal circles; show that the triangle PZQ is self-conjugate with respect to any of the circles.

(Poncelet, *Prob. Proj.*, vol. i., p. 43.)

Ex. 187. The centres of similitude of two circles and the four points in which the line of centres meets the circles form a system in involution.

Ex. 188. The poles of the radical axis of two circles with respect to these circles divide harmonically the straight line which joins their centres of similitude.

Ex. 189. If P and Q be any two points in an arc of the circumscribed circle of a triangle ABC external to the triangle, and if through P and Q circles be described touching the sides of the triangle internally in D, E, and F, show that AD, BE, and CF are concurrent.

Ex. 190. If the circles in Ex. 189 be described so as to touch the sides externally in D_1, E_1, and F_1, show that D_1, E_1, and F_1 are collinear.

Ex. 191. Using the notation of Ex. 189 and 190, show that

i. AD_1, BE_1, and CF are concurrent;
ii. AD_1, BE, and CF_1 are concurrent;
iii. AD, BE_1, and CF_1 are concurrent.

Ex. 192. With the same notation, prove

i. D, E, and F_1 collinear;

ii. D, E_1, and F collinear;

iii. D_1, E, and F collinear.

(*Nouv. Annales*, vol. x.)

Ex. 193. ABC are three collinear points and O any other point; circles are described through OAB, OAC, and OBC; P and Q are any two points between A and C, and through P and Q are drawn circles touching OAB, OAC, and OBC in D, E, and F on the same side of AC as O and in D_1, E_1, and F_1 on the opposite side; show that the circles DEF, D_1E_1F, D_1EF_1, and DE_1F_1 pass through O.

Ex. 194. Show that the circles $D_1E_1F_1$, DEF_1, DE_1F, and D_1EF also pass through the same point.

Ex. 195. Describe a circle which shall touch two given circles and pass through a given point. (By inversion.)

Ex. 196. Describe a circle which shall touch three given circles which pass through the same point. How many such circles are there? (By inversion.)

Ex. 197. The nine-point circle touches the inscribed and escribed circles of the triangle. (Feuerbach, 1822.)

[Let D and D_1 be the points in which the inscribed and an escribed circle touch one side of a triangle internally and externally; take H the mid-point of DD_1 as centre and HD^2 as constant of inversion. These circles are their own inverses, and the nine-point circle inverts into their fourth common tangent. For a proof by Elementary Geometry, see Baltzer, *Elemente der Mathematik*, 6th ed., vol. ii., p. 92.]

Ex. 198. If any point O in the triangle ABC be joined to the vertices, and if from O perpendiculars be drawn to OA, OB, and OC, show that they meet the sides a, b, and c in three collinear points. (This is the polar reciprocal of Ex. 26.)

Ex. 199. AB and CD are two equal links, BC and AD are two others equal to each other but longer than the former pair; they are placed so that BC and AD are the diagonals of a quadrilateral, of which AB and CD are two opposite sides. O, P, and Q are the middle points of AB, BC, and AD respectively; show that if O be fixed, P and Q are inverse points with respect to it.

Ex. 200. In what manner must a fifth link be fixed to P that Q may describe a straight line?

(Prof. Hart, in Kempe's *How to draw a Straight Line*, p. 19.)

QUESTIONS FROM THE EXAMINATION PAPERS

SET FOR GEOMETRY HONOURS GRADE IN THE LEAVING CERTIFICATE EXAMINATIONS FOR SCOTCH SCHOOLS.

1893.

6. If a straight line cuts the sides BC, CA, AB of a triangle in D, E, F respectively, prove that

$$BD \cdot CE \cdot AF = CD \cdot AE \cdot BF.$$

The straight line AB is bisected in X, AX is bisected in Y, CA is bisected in Y′, CY′ in X′; prove that XX′, YY′, BC are concurrent.

7. Define a centre of similitude of two circles.

Show that the orthocentre of a triangle is a centre of similitude of the nine-point and circumscribed circles, and find the other centre of similitude.

8. From the centre O of a fixed circle a perpendicular OP is drawn to a fixed straight line, and OQ taken in OP, or OP produced, such that the rectangle OP . OQ is equal to the square on the radius. Prove that any straight line through Q is harmonically divided by the given circle and the given line.

9. What is meant by the inverse of a given curve? Prove that the inverse of a circle is in general a circle. Mention an apparent exception.

By inversion, or otherwise, show how to describe a circle touching three given circles that intersect in one point.

10. Show that the locus of a point from which the tangents to two given circles are equal is a straight line at right angles to the line joining the centres.

What does this proposition become when (*a*) one of the circles reduces to a point, (*b*) both reduce to points?

1894.

6. If through a point O in the plane of a triangle ABC straight lines AO, BO, CO be drawn meeting the sides of the triangle in P, Q, R, then $\frac{BP}{CP} \cdot \frac{CQ}{AQ} \cdot \frac{AR}{BR} = -1$.

Enunciate the converse theorem, and show that its truth necessarily follows from that of the theorem.

Hence show that the medians of a triangle are concurrent.

8. When do four straight lines form a harmonic pencil?

If three such lines be cut in P, Q, R by a line parallel to the fourth, show that $PQ = QR$, and hence show how to find the fourth ray of a harmonic pencil when three are given.

9. Define a pole and a polar with respect to a given circle, and show how to construct a triangle, each of whose sides is the polar of the corresponding vertex with respect to a given circle.

10. What is meant by inversion?

Show that concentric circles invert into co-axal circles, and find the radical axis.

1895.

4. If a transversal meet the sides of a triangle BC, CA, AB in P, Q, R respectively, prove that

$$BP \cdot CQ \cdot AR + PC \cdot QA \cdot RB = 0.$$

ABCD is a parallelogram, X, Y are points on AB, CD such that XY is parallel to BC, and Z, W are points on BC, AD such that WZ is parallel to AB; prove that WX, ZY intersect on the diagonal BD, and WY, ZX on AC.

5. Prove that the three perpendiculars from the vertices of a triangle on the opposite sides meet in a point.

A point O is taken within a triangle ABC, and the lines AO, BO, CO cut the opposite sides in P, Q, R; if the rectangles AO . OP, BO . OQ, CO . OR be equal, show that O is the orthocentre.

6. Prove that any straight line, drawn from a given point to cut a given circle, is divided harmonically by the circle, and the polar of the point.

Hence show, assuming the harmonic properties of a complete quadrilateral, how to draw the polar of a given point with respect to a given circle by means of the ruler alone.

7. Define the radical axis of two circles, and show how to find its position. Prove that the square on the tangent, drawn to the one circle from a point on the other, is proportional to the distance of that point from the radical axis.

8. Define a centre of similitude of two circles. How many are there? Show that the line joining the points of contact of any circle that touches two fixed circles will pass through one of two fixed points, and distinguish the cases.

9. If the base of a triangle and its circumscribing circle be fixed, prove that the centres of the inscribed and the three escribed circles lie on two fixed circles that pass through the extremities of the base, and cut each other orthogonally.

1896.

4. Explain the convention of signs in dealing with line segments, and prove that if A, B, C, D be four collinear points, then

$$AB \cdot CD + BC \cdot AD + CA \cdot BD = 0.$$

5. What is meant by the centroid of a triangle? G is the centroid of the triangle ABC; O is any point; AQ and QR are respectively equal to, parallel to, and in the same direction as OB, OC; prove that OR lies along OG, and is equal to three times OG in length.

7. When do four points form a harmonic range?

If A, B, C, D be four such points in order, prove that any circle through A and C is cut orthogonally by the circle on BD as diameter.

8. Define the radical axis of two circles, and prove that in general the radical axes of three circles taken two and two are concurrent, stating carefully the special cases that may occur.

A circle passes through two fixed points, and cuts a given circle; prove that the common chord passes through a fixed point.

9. Prove that in general two circles have two centres of similitude, and state the exception.

Prove also that two squares with their sides parallel have, in general, two centres of similitude.

10. Define the polar of a given point with respect to a given circle.

The radius through a point P, inside a circle, cuts its polar in Q; any line through P cuts the circle in XY; prove that XQ, YQ are equally inclined to PQ.

1897.

4. Show that any transversal parallel to one ray of a harmonic pencil has equal parts intercepted between the other three.

Three collinear points, A, B, C being given, determine a point D that, with them, completes a harmonic range; and show clearly how many solutions there are.

5. From a point P on the circumscribing circle of a triangle PX, PY, PZ are drawn perpendicular to the respective sides BC, CA, AB. Prove that X, Y, Z are collinear.

Show that the line YZ is greatest, for all positions of P, when AP is a diameter.

6. If I be the inscribed, and J an escribed centre of a triangle ABC, and AIJ cuts the side BC in P and the circumscribing circle in X, prove that $XI^2 = XJ^2 = XP \cdot XA$.

Hence, or otherwise, show that if HIK, LJM be drawn parallel to BC to cut AB in H, L, and AC in K, M, the circles on HL, KM as diameters touch in X.

7. Prove that the three perpendiculars of a triangle meet in a point.

Prove that the line joining this point to the mid-point of any side, and produced to meet the circumscribing circle, is bisected where it meets the side.

8. Define the limiting points of a system of co-axial circles.

If these are given, show how to find the two circles of the system that have a given radius.

9. Explain the principle of Inversion.

Prove that if the nine points circle be inverted with reference to the mid-point of one side, the inverse will be a straight line antiparallel to that side with respect to the other two.

1898.

2. What is meant by the cross ratio of the range (ABCD)?

Show that the 24 cross ratios, obtainable from four collinear points, A, B, C, D, reduce four and four, in general, to only six different ratios; and show that, if $(ABCD) = \frac{1}{2}$, they reduce to only three.

3. If a transversal cut the sides BC, CA, AB of a triangle in P, Q, R respectively, prove that

$$BP \cdot CQ \cdot AR = CP \cdot AQ \cdot BR.$$

In the sides BA, AC of a triangle points P, Q are taken so that $2BP = 2CQ = AB + AC$, prove that PQ bisects the base.

4. Prove that lines intersected by parallel planes are cut proportionally. ABCD is a non-planar quadrilateral; AB and CD are divided in P, Q, so that $AP : PB = DQ : QC$; and BC, AD in R, S, so that $BR : RC = AS : SD$; prove PQ, RS intersect and divide each other respectively in the some ratios as those in which AD, AB are divided.

Alternative Questions.

5*a*. Prove that the inverse of a straight line is, in general, a circle.

Describe, and prove the principle of, the "Peaucellier cell," or "inverting linkage."

Or

5*b*. If A, B be fixed points on a circle and P, Q any diameter, prove that PA, BQ intersect on a circle cutting the given circle orthogonally.

6*a*. Define the centres of similitude of two circles: and examine their position when one of the circles becomes a straight line.

From O, a centre of similitude of two given circles a straight line OPQ is drawn to cut them in the non-corresponding points, P, Q; any circle through P, Q cuts them again in R, S; prove that O, R, S are collinear.

Or

6*b*. A, B, C, D are four collinear points; prove that a single collinear point O can in general be found so that $OA \cdot OB = OC \cdot OD$.

Find the locus of points at which AC, BD subtend equal angles.

7*a*. Prove that the centroid of a triangle trisects the distance between the orthocentre and the circumcentre.

Or

7*b*. What is meant by a system of co-axial circles?

Having given a circle, and, as axis, a straight line which does not cut it, show how to construct two co-axial circles of given radius.

1899.

2. If a transversal cut the sides BC, CA, AB of a triangle in P, Q, R respectively, prove that

$$BP \cdot CQ \cdot AR = CP \cdot AQ \cdot BR,$$

showing how the signs of the segments must be taken.

If the triangle be equilateral, Q within CA, R without AB, and $CQ = RA$, show that the triangles CPQ, CQB are similar.

3. When is a triangle said to be self-conjugate to a given circle? Prove that such a triangle has one angle within the circle and that angle obtuse.

A triangle being given, find the circle to which it is self-conjugate.

5*a*. State and prove the fundamental harmonic property of a complete quadrilateral.

Hence, or otherwise, prove that the mid-points of its diagonals are collinear.

Or

5*b*. If the lines joining the vertices of two triangles are concurrent, the intersections of their corresponding sides are collinear.

If D, E, F are the points of contact of the inscribed circle with the sides BC, CA, AB of a triangle, prove that the intersections of the lines EF, BC; FD, CA; DE, AB are collinear. Explain your result if the triangle is equilateral.

6*a*. The equal sides of an isosceles triangle of constant form and size pass through two fixed points. Prove that the bisectors of the interior and exterior vertical angles pass each through a fixed point, and that the base touches a fixed circle.

Or

6*b*. ABC, DCB are triangles equal in all respects on the same base and on the same side of it; PQRS is a line parallel to the base cutting BA, BD, CA, CD in P, Q, R, S respectively. Prove that

$$PQ \cdot PR = BQ \cdot QD - BP \cdot PA.$$

7*a*. Define the radical axis of two circles, and prove its fundamental property.

If from any point tangents be drawn to the circles, prove that the difference of their squares is proportional to the distance of the point from the radical axis.

Or

7*b*. Prove that the inverse of a circle is in general a circle. Prove also that the centre of the inverse curve is the inverse of the point conjugate to the centre of inversion with respect to the first circle.

Show that one point can in general be found such that three circles invert, with respect to it, each into itself.

1900.

3. Define the radical axis of two circles by a definition that includes all cases, and prove that the radical axes of three circles taken two and two are concurrent.

Hence, or otherwise, prove that if a variable circle pass through two fixed points and cut a fixed circle, their common chord passes through a fixed point.

4. Define Inverse Points.

A circle whose centre is A inverts into a circle whose centre is B with O as centre of inversion, and P is the inverse point of B. Show that if A be taken as the centre of inversion, O and P are inverse points with regard to the first mentioned circle.

Alternative Questions.

5*a*. Define a harmonic range, and show how to find the fourth point when three points are given.

Carefully examine the number of solutions.

Or

5*b*. If ACBD be a harmonic range and O be the mid-point of AB, prove that $OC \cdot OD = -OA \cdot OB$, and show that the same statement is true if CD be bisected instead of AB.

Prove that any circle on AB (or CD) as diameter is cut orthogonally by all circles through C, D (or A, B).

6*a*. If a point P be taken in the plane of a triangle ABC and AP, BP, CP meet BC, CA, AB in D, E, F, show that

$$BD \cdot CE \cdot AF + CD \cdot AE \cdot BF = 0.$$

In a triangle ABC, AB is produced one-fourth of its length to N, and AC is produced two-thirds of its length to M; BM and CN meet in O, and AO meets BC in L; show that BL is equal to half LC.

Or

6*b*. State and prove the relation existing between the segments into which any transversal divides the sides of a triangle.

From the mid-point of the base BC of an isosceles triangle ABC a straight line is drawn meeting AB, AC in P, Q; show that AB is a harmonic mean between AP and AQ.

7*a*. Define the polar of a point with regard to a circle, and determine it when the point is within the circle.

P, Q are two points and O is the centre of a given circle, PM, QN are the perpendiculars from P on the polar of Q, and from Q on the polar of P with respect to this circle, show that

$$PM : QN = OP : OQ.$$

Or

7*b*. Parallel diameters of two circles are drawn, prove that the straight lines joining their extremities pass through one of two fixed points on the line joining their centres.

If a third circle touch the two given circles, prove that the line joining the points of contact passes through one or other of these two fixed points.

1901.

2. Explain how line segments are added.

If A, B, C, D are fixed collinear points, and O is a variable point in the same straight line, and X is another point in the line, such that

$$4OX = OA + OB + OC + OD,$$

show that the position of X is independent of that of O.

5*b*. Define the centre of similitude of two circles, and prove its fundamental property.

Prove that the centres of similitude of three circles are the six vertices of a complete quadrilateral.

6*a*. If P, Q, R be points in the sides BC, CA, AB of a triangle ABC, such that $BP \cdot CQ \cdot AR = CP \cdot AQ \cdot BR$, prove that P, Q, R are collinear.

Or

6*b*. If through any point O in the plane of a triangle ABC straight lines AO, BO, CO be drawn to the vertices

meeting the opposite sides of the triangle in L, M, N, then

$$BL \cdot CM \cdot AN + CL \cdot AM \cdot BN = 0.$$

If two triangles have their vertices two and two on three concurrent straight lines, prove that the intersections of the three pairs of corresponding sides are collinear.

7*a*. Define a harmonic range and a harmonic pencil.

Prove that if Q, P, A, B, C, D be six points in a circle such that P(ABCD) is a harmonic pencil, then Q(ABCD) is also a harmonic pencil.

ABCD is a quadrilateral inscribed in a circle, and AC passes through the intersection of the tangents at B and D; if P be any point in the circumference, prove that P(ABCD) is a harmonic pencil.

Or

7*b*. If any number of points are collinear, prove that their polars with respect to any circle are concurrent.

If four points form a harmonic range, prove that their polars with respect to any circle form a harmonic pencil.

1902.

4. Define a Harmonic Range, and show that if C and D are harmonic conjugates with respect to A and B, then A and B are harmonic conjugates with respect to C and D.

A circle is described touching a straight line AB in C, and having its diameter a harmonic mean between AC and CB; prove that it touches the circle on AB as diameter.

5*b*. Explain the principle of inversion, and prove that the inverse of a circle is, in general, a circle.

Prove that a system of non-intersecting co-axal circles inverts into a system of concentric circles when either of the limiting points of the co-axal system is taken as the centre of inversion.

6*a*. If any transversal meet the sides BC, CA, AB of a triangle in D, E, F respectively, then $AF \cdot BD \cdot CE = BF \cdot CD \cdot AE$.

Prove that the tangents drawn to a circle at the vertices of an inscribed triangle meet the opposite sides of the triangle in three collinear points.

7*a*. Define the polar of a given point with regard to a given circle.

A is a point within a circle, and AB is drawn from A perpendicular to the polar of A, meeting it in B; if PAQ be any chord through A, show that

$$PB : BQ = PA : AQ.$$

1903.

5*a*. If a transversal meet the sides BC, CA, AB of a triangle ABC in L, M, N respectively, prove that

$$BL \cdot CM \cdot AN = CL \cdot AM \cdot BN.$$

The inscribed circle of a triangle IJK touches IJ, JK in P, Q respectively, and PQ meets IK produced in R; express KR in terms of the sides of the triangle.

Or

5*b*. If straight lines through the vertices A, B, C of a triangle ABC be concurrent in O, and meet the opposite sides in L, M, N respectively, prove that

$$BL \cdot CM \cdot AN = LC \cdot MA \cdot NB.$$

If a point P be joined to A, B, C, and straight lines AD, BE, CF be drawn so that the angles BAD, CBE, ACF are respectively equal to PAC, PBA, PCB, prove that AD, BE, CF are concurrent.

6*a*. Define a harmonic range. P is any point in a circle whose diameter AB is produced to D; if PC be drawn to meet AB in C, so that the angle BPC be equal to the angle BPD; prove that A, C, B, D form a harmonic range.

Or

6*b*. A, B, C, D are four points in a straight line; prove that

$$AB \cdot CD - AC \cdot BD + AD \cdot BC = 0,$$

due regard being paid to the signs of the segments. Hence prove that the sum of the cross ratios (ABDC) and (ADBC) is unity.

7*a*. Define the radical axis of two circles, and prove that the radical axes of three circles taken two at a time meet in a point.

On the sides AB, AC of a triangle ABC segments of circles are described towards the triangle containing angles supplementary to half the angle BAC. Any circle through B and C meets these segments in M and N; prove that BM, CN meet on the bisector of A.

Or

7*b*. If any number of points are collinear, prove that their polars with respect to any circle are concurrent.

ABC is a triangle, and the tangent at A to the circumscribing circle meets BC in D and is produced to E, so that DE is equal to AD; prove that the polar of E with regard to any circle through B and C passes through A.

GEOMETRICAL CONICS.

INTRODUCTION.

The three curves known as the conic sections appear to have been discovered by Menaechmus, a geometer of the school of Plato, in the fourth century before the Christian era. It is probable that the discovery was made in attempting to solve the two famous problems of the trisection of the angle and the duplication of the cube, both of which can be solved by the use of the conic sections (see Ex. 35, 36, and 61). They may thus at first have been regarded as plane loci. It was soon, however, discovered that the section of a cone by a plane at right angles to its side is a parabola, an ellipse, or a hyperbola, according as the vertical angle of the cone is right, acute, or obtuse.

Euclid (about 300 B.C.) wrote a treatise on the conics in four books, which has been lost.

Archimedes (287-212 B.C.) also wrote on the subject. He discovered that the area of a parabolic segment is two-thirds of the triangle contained by its chord and the tangents at its ends, a discovery of great interest, as being the first real instance of the quadrature of a curvilinear area.

Apollonius, who flourished somewhat later than Archimedes, was the greatest writer on the conic sections in ancient times. His treatise contained eight books, the first four of which were founded on Euclid's earlier work. He showed that the three curves may be obtained from the same cone, whatever may be its vertical angle, by varying the inclination of the cutting plane to the side of the cone. (For an account of the contents of this work, see Taylor, *Ancient and Modern Geometry of Conics*, pp. xlii.-xlix.)

The names parabola, ellipse, and hyperbola given to the conic sections by Apollonius suggest the ideas of equality, defect, and excess, and were applied to the particular curves because in the parabola the square on any ordinate is *equal to* the rectangle contained by the latus rectum and the abscissa (Th. 5 iv.), while in the ellipse it is *less than*, and in the hyperbola *greater than*, that rectangle (Th. 15 iv.). [Compare the use of these words in Euclid VI. 27, 28, 29.]

Amongst the moderns we may name Kepler (1571-1631) and Newton (1642-1727), who gave a fresh interest to the study of the conics by showing that the orbits of the heavenly bodies are conic sections.

A list of writers of treatises on the conic sections up to 1833 will be found in the article "Conic Sections" by Prof. William Wallace in the seventh edition of the Encyclopædia Britannica.

The ancients deduced the properties of the conic sections

from those of the cone, making great use of the relation between the ordinate, abscissa, and latus rectum above mentioned.

Most recent writers who begin with the cone use the following theorem due to MM. Quetelet and Dandelin (1822): If a sphere be inscribed in a cone, and if a plane be drawn to touch the sphere, the point of contact is a focus of the conic section in which the plane cuts the cone; and the straight line in which the tangent plane cuts the plane of contact of the sphere and cone is the corresponding directrix.

We may also treat the curves as plane loci, as has been done in this work, deducing all their other properties from the constancy of the ratio of the distances of any point in a conic from the focus and directrix. This property was known to Pappus (about 300 A.D.), who made no special use of it. The existence of the foci of the ellipse and hyperbola was known to Apollonius, but he did not detect that of the parabola, the first mention of which is found in Pappus. The name focus, however, was first used by Kepler. The first treatise in which this property was used to define the conic sections was that of Boscovich (Rome, 1754).

GEOMETRICAL CONICS.

THE GENERAL CONIC.

Def. If a point move in a plane so that its distance from a given point is in a constant ratio to its distance from a given straight line, its locus is called a *conic section*, or simply a *conic*.

Def. The given point is called the *focus* (denoted throughout by S or S_1).

Def. The given straight line is called the *directrix*.

Def. The constant ratio is called the eccentricity (denoted throughout by e).

Thus if P be any point on a conic and M its projection on the directrix, we have from these definitions $SP = ePM$.

Def. The conic is called a *parabola*, an *ellipse*, or a *hyperbola*, according as e is equal to, less than, or greater than unity.

Def. The straight line through the focus perpendicular to the directrix is called the *axis*; it meets the conic in points called the vertices (denoted by A and A_1).

Def. A *tangent* to a conic is the limiting position of a secant when the points in which the latter cuts the curve become infinitely near each other.

Def. The *normal* at any point is the perpendicular to the tangent at that point.

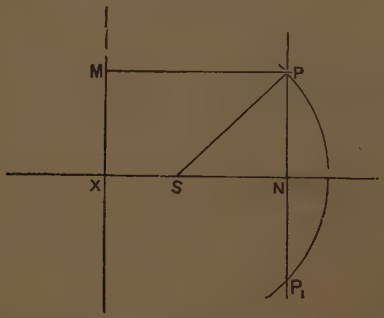
Def. The *ordinate* and *abscissa* of any point in a conic are the perpendicular from that point to the axis and the segment of the axis intercepted by that perpendicular and the vertex.

Def. The double ordinate through the focus is called the *latus rectum*, or the *parameter*.

Def. The projections on the axis, of the tangent and normal at any point, are called the *subtangent* and *subnormal*.

PROBLEM 1.

Given the focus, directrix, and eccentricity of a conic, to construct any number of points in it.



Draw the axis, meeting the directrix in X. In it take any point N. Through N draw a perpendicular to the axis.

With centre S and radius eXN describe a circle cutting this perpendicular in P and P_1.

P and P_1 are points in the conic.

Join SP, and draw PM perpendicular to the directrix.

$SP = eXN = ePM$. *Const. and I.* 34.

$\therefore$ P is a point in the conic. *Def.*

Similarly P_1 is in it.

In like manner by varying the position of N any number of pairs of points can be constructed.

Cor. 1. Every conic is symmetrical about its axis.

Cor. 2. If $e = 1$.

Bisect SX in A (not shown in the figure).

A is a vertex of the conic, since $SA = AX$. *Def.*

If N lie to the left of A, $eXN < SN$;

$\therefore$ the circle does not cut the perpendicular;

$\therefore$ no part of the curve lies to the left of A.

But if N lie to the right of A, $eXN > SN$;

$\therefore$ the circle always cuts the perpendicular.

Hence a parabola is a curve consisting of one infinite branch, and lying wholly on the side of the vertex remote from the directrix.

Cor. 3. If $e < 1$.

Divide SX internally in A and externally in A_1 in the ratio $e : 1$.

A and A_1 are the vertices of the conic. *Def.*

A_1 lies to the right of S, since $SA_1 < A_1X$.

If N lie outside the segment AA_1,

$$\frac{SN}{XN} < \frac{SA}{XA} < e; \quad \textit{Mod. Geom., Prob. 1.}$$

$$\therefore \quad eXN < SN;$$

$\therefore$ the circle does not cut the perpendicular;

$\therefore$ no part of the curve lies to the left of A, or to the right of A_1.

In like manner it may be shown that, when N lies in AA_1, the circle always cuts the perpendicular.

Hence an ellipse is a closed curve lying wholly between its vertices.

Cor. 4. If $e>1$.

As in Cor. 3, it may be shown that a hyperbola has two vertices, and that it is a curve consisting of two infinite branches lying wholly on the sides of its vertices remote from the directrix.

Cors. 2, 3, and 4 may be omitted on a first reading. To master them thoroughly the student should draw a few accurate figures, taking such simple numerical values for e as $\frac{1}{2}$, $\frac{1}{3}$, 2, 3, &c.

Ex. 1. The locus of the centres of all circles which touch a given straight line and pass through a given point is a parabola.

Ex. 2. The locus of the centres of all circles which touch a given straight line and a given circle is a parabola. Find its focus and directrix.

Ex. 3. According as a point is within, on, or without a parabola, its distance from the focus is less than, equal to, or greater than its distance from the directrix; and conversely.

Ex. 4. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the ellipse is within and the hyperbola without the parabola.

Ex. 5. Any straight line through the focus of a conic is divided harmonically by the curve and the directrix.

Ex. 6. The focus of a conic and its directrix are pole and polar with respect to the conic.

Ex. 7. To construct a conic having given two points in it and its focus and eccentricity.

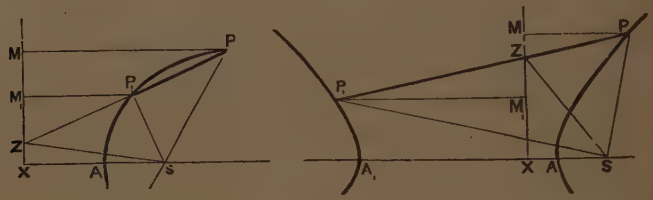
(Newton, *Principia*, Bk. I., Prop. 20.)

Ex. 8. To construct a conic having given two points in it and its directrix and eccentricity.

Ex. 9. Two different conics cannot have the same focus, directrix, and eccentricity.

THEOREM 1.

If a secant PP_1 *cut a conic in* P *and* P_1 *and the directrix in* Z, SZ *is the internal or external bisector of the angle* PSP_1.



Draw PM and P_1M_1 perpendicular to the directrix.

$\triangle$s PMZ, P_1M_1Z are similar; *VI.* 4.

$$\therefore \quad \frac{PM}{P_1M_1} = \frac{PZ}{P_1Z};$$

$$\therefore \quad \frac{SP}{SP_1} = \frac{e \,.\, PM}{e \,.\, P_1M_1} = \frac{PZ}{P_1Z};$$

$\therefore$ SZ is the internal or external bisector of the angle PSP_1. *VI.* 3 *or A.*

Cor. 1. If P and P_1 become coincident, $\angle$ PSZ becomes a right angle. Hence, if the tangent at P meet the directrix in Z, the angle PSZ is a right angle; and conversely.

Cor. 2. A tangent may thus be drawn to a conic at any point in the curve, or from any point in the directrix.

Cor. 3. Tangents to a conic at the ends of a focal chord meet in the directrix.

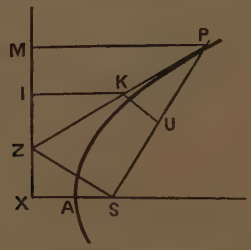
Ex. 10. A straight line cannot cut a conic in more points than two.

Ex. 11. If the focus and two points in a conic be given, the directrix must pass through one or other of two fixed points.

Ex. 12. If the focus and three points in a conic be given, construct the directrix. How many cases are there?

THEOREM 2.

If from any point K *in the tangent to a conic at* P *perpendiculars be drawn,* KI *to the directrix and* KU *to* SP, SU = *e*KI.



Let the tangent meet the directrix in Z.

Join SZ, and draw PM perpendicular to the directrix.

$\angle$ PSZ is a right angle. *Th.* 1, *Cor.* 1.

$\angle$ PUK is a right angle. *Hyp.*

$\therefore$ KU is parallel to SZ; *I.* 28.

$$\therefore \frac{SU}{SP} = \frac{ZK}{ZP}. \qquad \textit{VI.}\ 2.$$

Now $\triangle$s ZKI, ZPM are similar; *VI.* 4.

$$\therefore \frac{ZK}{ZP} = \frac{KI}{PM}.$$

$$\therefore \frac{SU}{SP} = \frac{KI}{PM}.$$

But SP = *e*PM; *Def.*

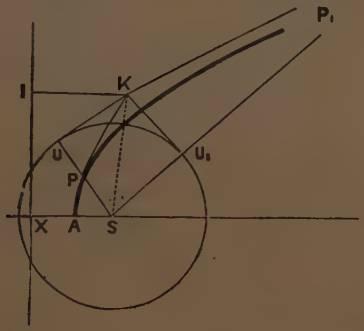
$\therefore$ SU = *e*KI.

Conversely: If SU = *e*KI, K *is a point in the tangent at* P.

[This theorem is attributed to Prof. J. C. Adams.]

PROBLEM 2.

To draw a pair of tangents to a conic from a given external point.



Let K be the given point.

Draw KI perpendicular to the directrix.

With centre S and radius eKI describe a circle.

Draw KU and KU_1 tangents to the circle.

Let SU and SU_1 meet the curve in P and P_1.

Join KP and KP_1 : these are the required tangents.

KU is perpendicular to SP ; *III.* 18.

$SU = eKI$; *Const.*

$\therefore$ KP is a tangent to the conic. *Th.* 2.

Similarly KP_1 is a tangent.

Cor. 1. A pair of tangents to the same branch of a conic subtend equal angles at the focus.

Cor. 2. A pair of tangents, one to each branch of a hyperbola, subtend supplementary angles at the focus.

Ex. 13. If P be a point without a conic, and M its projection on the directrix, SP is greater than ePM; and conversely.

Ex. 14. If a tangent to a conic meet the directrix in Z and the latus rectum in K, SK = eSZ.

Ex. 15. If K be taken on the other side of the directrix in the figure of Prob. 2, what change must be made in the construction?

Ex. 16. There cannot be drawn more than two tangents to a conic from a given point.

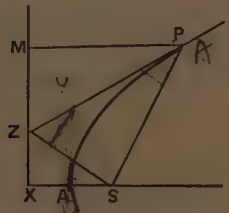
Ex. 17. OQ and OQ_1 are fixed tangents to a conic, and RPR_1 is any third tangent cutting OQ in R and OQ_1 in R_1; show that the angle RSR_1 is constant.

Ex. 18. If a quadrilateral be described about a conic, a pair of opposite sides subtend supplementary angles at the focus.

THE PARABOLA.

THEOREM 3.

The tangent to a parabola at any point P *bisects the angle* SPM, M *being the projection of* P *in the directrix.*



Let the tangent at P meet the directrix in Z. Join SZ.

$\angle$ PSZ is a right angle; *Th.* 1, *Cor.* 1.

$\therefore$ $SZ^2 = PZ^2 - SP^2 = PZ^2 - PM^2 = MZ^2$; *I.* 47.

$\therefore$ SZ = MZ.

$\therefore$ $\triangle$s SPZ, MPZ are congruent; *I.* 8.

$\therefore$ $\angle$ SPZ = $\angle$ MPZ.

Converse: *The straight line which bisects the angle* SPM *is the tangent at* P.

Cor. 1. The tangent at the vertex is perpendicular to the axis.

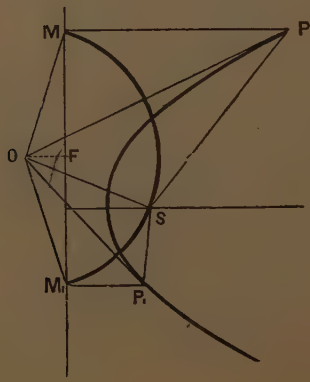
Ex. 19. Tangents at the ends of a focal chord intersect at right angles on the directrix: and a pair of tangents which meet at right angles, meet on the directrix.

Ex. 20. Two parabolas which have the same focus and axis, but their vertices on opposite sides of the focus, cut each other orthogonally.

Ex. 21. If a parabola roll on an equal parabola, their vertices having been orginally coincident, the focus of the former traces out the directrix of the latter.

PROBLEM 3.

To draw a pair of tangents to a parabola from an external point.



Let O be the given point.

With centre O, and radius OS, describe a circle cutting the directrix in M and M_1.

From M and M_1 draw perpendiculars to the directrix meeting the curve in P and P_1.

Join OP and OP_1; these are the required tangents.
Join OM, OM_1, SP, and SP_1.

$\triangle$s SOP, MOP are congruent; *I.* 8.

$\therefore \angle SPO = \angle MPO$;

$\therefore$ OP is the tangent at P. *Th.* 3.

Similarly OP_1 is the tangent at P_1.

[The general method of Prob. 2 may also be used.]

Cor. A pair of tangents to a parabola have equal projections on the directrix.

Draw OF perpendicular to MM_1.

$MF = FM_1$. *III.* 3.

THEOREM 4.

If OP *and* OP_1 *be a pair of tangents to a parabola, the triangles* PSO *and* OSP_1 *are similar.*

(See figure of Prob. 3.)

$\angle PSO = PMO = P_1M_1O = P_1SO$. *I.* 8, *&c.*

Now PMM_1P_1S is a pentagon;

$\therefore$ its angles = 6 right angles. *I.* 32.

But the angles at M and M_1 are right angles.

$\therefore$ the other three angles = 4 right angles.

Now each of these three angles is bisected. *Th.* 3.

Take one half of each,

$\therefore \angle SPO + \angle OSP + \angle SP_1O = 2$ right angles.

But $\angle SPO + \angle OSP + \angle SOP = 2$ right angles. *I.* 32.

$\therefore \angle SOP = SP_1O$.

$\therefore$ $\triangle$s PSO, OSP_1 are similar. *VI.* 4.

Ex. 22. OQ and OQ_1 are a pair of tangents to a parabola, a third tangent RPR_1 cuts OQ in R and OQ_1 in R_1; show that $QR : RO :: OR_1 : R_1Q_1$.

Ex. 23. Hence when RPR_1 is parallel to QQ_1, show that OQ and OQ_1 are bisected.

Ex. 24. If a parabola touch the sides of the triangle ABC in D, E, and F respectively, AD, BE, and CF are concurrent. (Use Prob. 3, Cor., and Ceva's Theorem.)

Ex. 25. If OQ and OQ_1 be a fixed pair of tangents and RPR_1 a variable tangent cutting OQ in R and OQ_1 in R_1, show that the angle RSR_1 is the supplement of the angle QOQ_1.

Ex. 26. If a parabola touch the three sides of a triangle its focus lies on the circumcircle of the triangle.

(Lambert, 1761.)

Ex. 27. Construct the focus of the parabola which touches two given straight lines in given points.

Ex. 28. If the sides of a variable right angle be tangents to a parabola, the locus of the vertex of the right angle is the directrix.

Ex. 29. The squares on tangents to a parabola from an external point are to each other as the focal distances of their points of contact.

THEOREM 5.

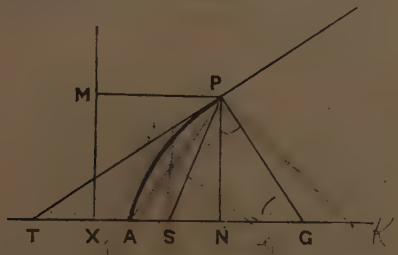
If the tangent and normal at any point P *in a parabola meet the axis in* T *and* G *respectively, and if* PN *and* AN *be the ordinate and abscissa of* P,

i. $ST = SP = SG$;

ii. *The subtangent* $TN = 2AN$;

iii. *The subnormal* $NG = 2AS$;

iv. $PN^2 = 4AS \cdot AN$.



Draw PM perpendicular to the directrix.

i. $\angle STP = \angle TPM = \angle SPT$. *I.* 29, *Th.* 3.

Also their complements $\angle SGP$ and $\angle SPG$ are equal.

$\therefore \quad ST = SP = SG$. *I.* 6.

ii. $ST = SP = PM = NX$. *I.* 34, *&c.*

$SA = AX$. *Def.*

$\therefore \quad AT = AN$; *Ax.* 3.

$\therefore \quad TN = 2AN$.

iii. $SG = SP = PM = XN$. *I.* 34, *&c.*

From SG and XN take away SN,

$\therefore \quad NG = SX = 2AS$.

iv. $PN^2 = TN \cdot NG = 4AS \cdot AN$. *VI.* 8, *Cor.*

Converse of part iv. *If* A *and* S *be fixed points and* PN *a variable ordinate to the straight line* AS, *and if* $PN^2 = 4AS \cdot AN$, *the locus of* P *is a parabola whose focus is* S *and vertex* A.

Cor. The latus rectum is equal to 4AS.

Part iv. may be proved independently, thus:

$$PN^2 = SP^2 - SN^2 = PM^2 - SN^2, \qquad I.\ 47.$$
$$= XN^2 - SN^2 = XS^2 + 2XS \cdot SN, \qquad I.\ 34 \text{ and } II.\ 4.$$
$$= 4AS^2 + 4AS \cdot SN = 4AS(AS + SN),$$
$$= 4AS \cdot AN.$$

Ex. 30. The diameter of the circle which passes through the vertex and the ends of the latus rectum of a parabola is equal to 5AS.

Ex. 31. If in PN the ordinate of a parabola a point Q be taken, such that QN : PN in a constant ratio the locus of Q is a parabola. Find its vertex and focus.

Ex. 32. If P be the point in a parabola such that SP is equal to the latus rectum, the triangle SPG is equilateral.

Ex. 33. If the ordinate of a point P bisect the subnormal of another point Q, the ordinate of P is equal to the normal of Q.

Ex. 34. The locus of the middle point of a focal chord of a parabola is a parabola. Find its vertex and focus.

Ex. 35. To find two mean proportionals between two given straight lines.

(Let AB and AC, the two given straight lines, be placed at right angles. Describe parabolas with the vertex A, and having AC and AB for tangents at their vertices and latera-recta equal to AB and AC. Let these parabolas intersect in P. Draw PM and PN perpendicular to AB and AC. PM and PN are the lines required.)

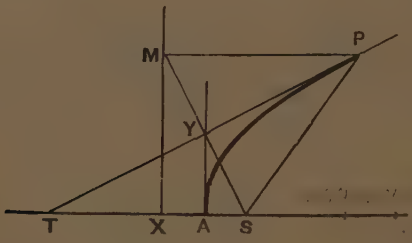
[This construction is attributed to Menaechmus. See Heiberg's *Archimedis Opera*, vol. iii. p. 96.]

Ex. 36. To construct a cube which shall be double a given cube.

Ex. 37. If the chord PP_1 of a parabola cut the axis in O, and if PN and P_1N_1 be the ordinates of P and P_1, show that AO is a mean proportional between AN and AN_1.

THEOREM 6.

The image of the focus of a parabola in any tangent lies in the directrix.



Let PT be the tangent at P.

Join SP.

Draw PM perpendicular to the directrix.

Join SM meeting PT in Y.

$\triangle$s SPY, MPY are congruent; *I.* 4.

$\therefore$ SY = YM, and PY perpendicular to SM;

$\therefore$ M is the image of S in PT;

$\therefore$ the image of S in any tangent lies in the directrix.

Cor. 1. Y is the middle point of TP; or, a perpendicular from the focus on a tangent bisects it.

Cor. 2. Y lies on the tangent at A; or, the locus of the foot of a perpendicular from the focus on a tangent is the tangent at the vertex.

(Newton, *Principia*, Bk. I., Lemma 14, Cor. 3.)

Cor. 3. SY is a mean proportional between AS and SP. (Newton, *Principia*, Bk. I., Lemma 14.)

Ex. 38. To draw a common tangent to two parabolas which have a common focus but not a common axis.

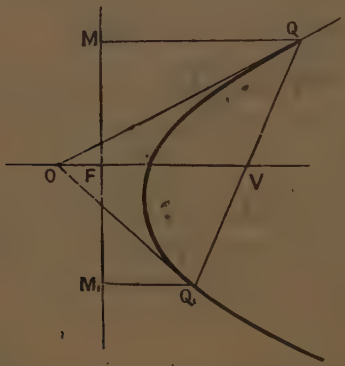
Ex. 39. Given the focus of a parabola which touches the sides of a triangle, to find its directrix.

Ex. 40. If one triangle can be inscribed in a given circle so that its sides touch a given parabola, any number of triangles can be so described.

Ex. 41. Perpendiculars from any point in the circumference of a circle to the sides of an inscribed triangle meet these sides in three collinear points. (Use Cor. 2, and Ex. 26.)

THEOREM 7.

The straight line parallel to the axis through the point of intersection of a pair of tangents to a parabola, bisects their chord of contact.



Let OQ, OQ_1 be a pair of tangents, with OV parallel to the axis meeting QQ_1 in V and the directrix in F.

$$\frac{QV}{VQ_1} = \frac{MF}{FM_1} \qquad \textit{VI.}\ 2.$$

But $MF = FM_1$; *Prob.* 3, *Cor.*

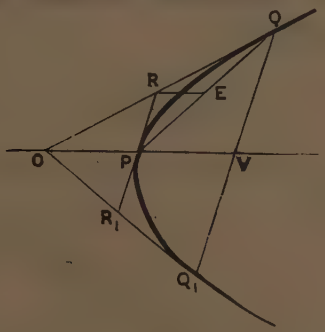
$\therefore\ QV = VQ_1$.

Conversely: Tangents at the end of a chord meet on the line parallel to the axis which bisects the chord.

Def. A straight line parallel to the axis of a parabola is called a *diameter*.

THEOREM 8.

A diameter of a parabola bisects all chords parallel to the tangent at the point where it cuts the curve.



Let OPV be any diameter, RPR_1 the tangent at P, and QVQ_1 any chord parallel to RR_1.

At Q draw the tangent QRO meeting the given tangent in R, and the given diameter in O.

Join QP.

Draw RE parallel to OV, meeting QP in E.

$$\frac{OP}{PV} = \frac{OR}{RQ} = \frac{PE}{EQ}. \qquad \textit{VI.}\ 2.$$

But $PE = EQ$; *Th.* 7.

$\therefore\ OP = PV.$

Similarly, it may be shown that the tangent at Q_1 meets VP in the same point O.

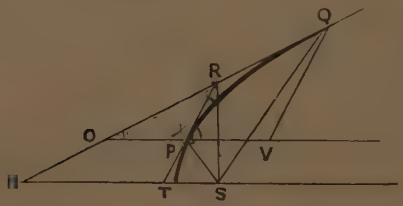
$\therefore\ QV = VQ_1.$ *Th.* 7.

Similarly, all chords parallel to RR_1 are bisected by the diameter through P.

Def. A straight line, such as QV, drawn from any point Q in a parabola parallel to RR_1 is called the *ordinate* of Q with respect to the diameter through P, and PV is the corresponding *abscissa*.

THEOREM 9.

If QV *be an ordinate to the diameter through* P, $QV^2 = 4SP \cdot PV$.



At Q draw a tangent, meeting the tangent at P in R, the given diameter in O, and the axis in H.

Let the tangent at P meet the axis in T.

Join SP, SQ, and SR.

$\angle SRP = \angle SQR$, *Th.* 4.

$= \angle SHR$ ($\because SQ = SH$.) *Th.* 5 i.

$= \angle POR$. *I.* 29.

$\angle SPR = \angle OPR$. *Th.* 3.

$\therefore$ $\triangle$s SPR, OPR are similar; *VI.* 4.

$\therefore$ SP : PR :: PR : PO.

But $PR = \frac{1}{2}QV$, and $OP = PV$. *Th.* 8, *&c.*

$\therefore SP : \frac{1}{2}QV :: \frac{1}{2}QV : PV$;

$\therefore QV^2 = 4SP \,.\, PV$.

Cor. If QSQ_1 be any focal chord, and PV the diameter which bisects it, $QSQ_1 = 4SP$.

Def. QSQ_1 (*Th.* 9 *Cor.*) is called the *parameter* of the diameter PV.

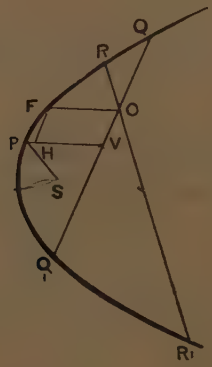
Ex. 42. The normals to a parabola at the ends of a focal chord intersect at right angles on the diameter which bisects the chord.

Ex. 43. If QD be drawn perpendicular to the diameter PV which bisects the chord QQ_1 in V, show that $QD^2 = 4AS \,.\, PV$.

Ex. 44. Show how to place in a parabola a focal chord of a given length.

THEOREM 10.

If two chords of a parabola intersect, the rectangles contained by their segments are to each other as the focal distances of the ends of the diameters which bisect the chords.



Let QOQ_1 and ROR_1 be two intersecting chords, and PV and P_1V_1 (the latter not shown in the figure) the diameters which bisect them; it is required to prove $QO \cdot OQ_1 : RO \cdot OR_1 :: SP : SP_1$.

Through O draw OF parallel to PV meeting the parabola in F.

Through F draw FH parallel to QQ_1 meeting PV in H.

Join SP.

$$\begin{aligned} QO \cdot OQ_1 &= QV^2 - OV^2, && II.\ 5. \\ &= QV^2 - FH^2, && I.\ 34. \\ &= 4SP \cdot PV - 4SP \cdot PH, && Th.\ 9. \\ &= 4SP(PV - PH), \\ &= 4SP \cdot HV, \\ &= 4SP \cdot FO. && I.\ 34. \end{aligned}$$

Similarly, $RO \cdot OR_1 = 4SP_1 \cdot FO.$

$$\therefore\ QO \cdot OQ_1 : RO \cdot OR_1 :: SP : SP_1.$$

If O be without the parabola use II. 6 instead of II. 5.

Ex. 45. Deduce Ex. 29 from Th. 10.

Ex. 46. If a parabola cut the sides of a triangle, or those sides produced, a in D and D_1, b in E and E_1, and c in F and F_1, show that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{CD_1}{BD_1} \cdot \frac{AE_1}{CE_1} \cdot \frac{BF_1}{AF_1}.$$

Ex. 47. Extend Ex. 46 to any polygon cut by a parabola.

Ex. 48. Using the notation of Ex. 46, show that if AD, BE, and CF are concurrent, so are AD_1, BE_1, and CF_1.

Ex. 49. Deduce from the preceding exercises that if a parabola touch the sides of a triangle in D, and E, F, AD, BE, and CF are concurrent.

Ex. 50. Given two of the points in which a parabola touches the sides of a triangle, to find the third.

Ex. 51. If a hexagon be inscribed in a parabola, the points of intersection of its opposite pairs of sides are collinear. (*See Mod. Geom., Th.* 5.)

Ex. 52. ABCDE is a pentagon inscribed in a parabola, the tangent at A meets CD in X, AB and DE meet in Y, and BC and EA meet in Z; show that X, Y, and Z are collinear.

Ex. 53. If a complete quadrangle be inscribed in a parabola, the tangents at any two of its vertices intersect on the sides of its diagonal triangle.

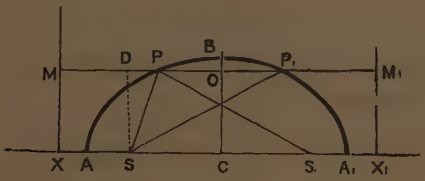
Ex. 54. If a complete quadrilateral be described about a parabola its diagonal trilateral coincides with the diagonal triangle of the complete quadrangle whose vertices are the points of contact.

Ex. 55. If a triangle be inscribed in a parabola, and if at the vertices tangents to the parabola be drawn, the points in which the tangents meet the opposite sides of the triangle are collinear.

THE ELLIPSE AND HYPERBOLA.

THEOREM 11.

The ellipse and hyperbola are symmetrical with respect to the straight line which bisects AA_1 *perpendicularly, and have therefore a second focus and directrix.*



(See also figure of Th. 12.)

Let PP_1 be any chord parallel to AA_1 meeting the directrix in M.

Bisect PP_1 in O, AA_1 in C, and join CO.

Draw SD perpendicular to PP_1.

Join SP and SP_1.

$$SP_1^2 - SP^2 = e^2(P_1M^2 - PM^2). \qquad \textit{Def.}$$

Also $SP_1^2 - SP^2 = P_1D^2 - PD^2$. *I.* 47.

$\therefore\ P_1D^2 - PD^2 = e^2(P_1M^2 - PM^2)$.

$$i.e.\ (P_1D + PD)(P_1D - PD) = e^2(P_1M + PM)(P_1M - PM),$$
$$i.e.\ (2.OD)(2.OP) = e^2(2.OM)(2.OP);$$
$$\therefore\ \frac{OD}{OM} = e^2.$$

But D is at a constant distance from the directrix, since SD is parallel to XM;

$\therefore$ O is also at a constant distance from it;

$\therefore$ the locus of O is the straight line which bisects AA_1 perpendicularly.

But P_1 is the image of P in CB;

$\therefore$ the curve is symmetrical with respect to CB.

Hence if CS_1, CA_1, and CX_1 be taken equal to CS, CA, and CX respectively, and if through X_1 there be drawn X_1M_1 perpendicular to the axis, S_1 and X_1M_1 are a focus and directrix by means of which the same curve may be described.

Cor. 1. Every chord through C is bisected at C.

Def. Such chords are called *diameters,* and C the *centre.* The ellipse and hyperbola are called *central conics.*

Cor. 2. $SS_1 = eAA_1$, $AA_1 = eXX_1$, and $CS \cdot CX = CA^2$.

For, from the definition and symmetry of the figure,

$$\begin{aligned} SS_1 &= S_1A \mp SA \qquad = SA_1 \mp SA \\ &= e(A_1X \mp AX) = e(AX_1 \mp AX) \\ &= eAA_1. \end{aligned}$$

[Read the upper signs for the ellipse and the lower

for the hyperbola in this and all the succeeding demonstrations.]

Similarly, $AA_1 = eXX_1$.

Hence, $CS = eCA$, and $CA = eCX$;

$\therefore \quad CS \cdot CX = CA^2$.

Def. AA_1 is called the *major* or *transverse axis.*

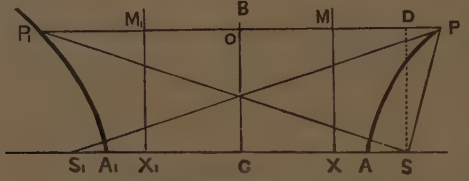
Def. In the ellipse, let BC produced meet the curve again in B_1, BB_1 is called the *minor* or *conjugate axis.*

Def. In the hyperbola, BB_1 does not meet the curve, but is taken of such a length that $CB^2 = CS^2 - CA^2$. It is still called the *minor* or *conjugate axis.* (See Th. 13.)

Def. There is a hyperbola having BB_1 for its major and AA_1 for its minor axes: it is called the *conjugate* of the given hyperbola.

THEOREM 12.

The sum, in the ellipse, and the difference, in the hyperbola, of the distances of any point on the curve from the foci is equal to the major axis.



[See also figure of Th. 11.]

Let P be any point on the curve.

Join SP and S_1P.

$$SP \pm S_1P = e(PM \pm PM_1)$$
$$= eMM_1$$
$$= eXX_1 \qquad I.\ 34.$$
$$= AA_1 \qquad Th.\ 11,\ Cor.\ 2.$$

Cor. $SP \pm S_1P$ is greater or less than AA_1 according as P lies on the convex or concave side of the curve; and conversely.

THEOREM 13.

In the ellipse $CB^2 = CA^2 - CS^2 = (1 - e^2)CA^2$.

In the figure of Th. 11, suppose SB and S_1B joined.

$\triangle$s SCB, S_1CB are congruent; *I.* 4.

$\therefore$ $SB = S_1B$.

But $SB + S_1B = AA_1$; *Th.* 12.

$\therefore$ $SB = CA$.

$\therefore$ $CB^2 = SB^2 - CS^2 = CA^2 - CS^2$. *I.* 47.

Now $CS^2 = e^2CA^2$; *Th.* 11, *Cor.* 2.

$\therefore$ $CB^2 = (1 - e^2)CA^2$.

Ex. 56. Two central conics which have the same centre but their corresponding axes at right angles to each other, cut each other (if at all) in four concyclic points.

Ex. 57. Find the locus of the centres of all circles which touch each of two given circles.

Examine the cases in which the circles are wholly external to each other, one wholly within the other, and cut each other.

Ex. 58. Hence describe a circle which shall touch each of three given circles.

Ex. 59. If a circle be regarded as an ellipse whose eccentricity is zero, find its foci and directrices.

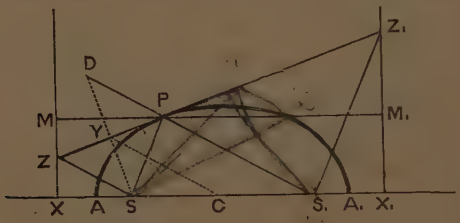
Ex. 60. If a pair of parallel straight lines be regarded as a hyperbola whose eccentricity is infinite, find its foci and directrices.

Ex. 61. To trisect a given rectilineal angle.

(Let O be the vertex of the angle; about O describe a circle cutting its sides in A and B; with A as focus and the bisector of $\angle AOB$ as directrix, describe a hyperbola whose eccentricity is 2. Let the hyperbola cut the circle in P and Q, OP and OQ trisect $\angle AOB$ and its conjugate.)

THEOREM 14.

The tangent to a central conic at any point P *makes equal angles with the focal distances of* P.



[See also figure of Th. 16.]

Let the tangent at P meet the directrices in Z and Z_1.

Join SZ and S_1Z_1.

Through P draw MPM_1 perpendicular to the directrices.

$\triangle$s MPZ, M_1PZ_1 are similar ; *VI.* 4.

$$\therefore \quad \frac{ZP}{Z_1P} = \frac{PM}{PM_1} = \frac{SP}{S_1P}.$$

Also $\angle$s PSZ, PS_1Z_1 are right angles. *Th.* 1, *Cor.*

$\therefore$ $\triangle$s PSZ, PS_1Z_1 are similar ; *VI.* 7.

$\therefore$ $\angle SPZ = S_1PZ_1$.

Cor. 1. The tangent at either vertex is perpendicular to the axis.

Cor. 2. The normal at P also makes equal angles with SP and S_1P.

Conversely: The straight line which makes equal angles with SP *and* S_1P *is either the tangent or normal at* P.

Def. The circle whose centre is either focus and whose radius is AA_1 is called the *director circle.* [In the limiting case of the parabola it coincides with the directrix.]

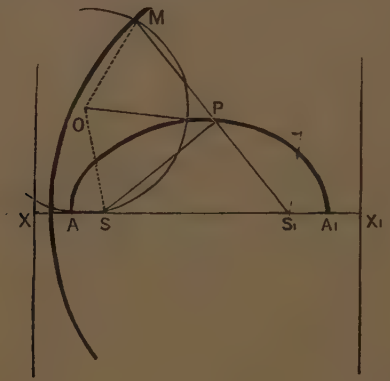
Note.—Boscovich (1754), who gave its name to the directrix, points out the analogy between this circle and the directrix. He, however, left it to some later writer to name it the *director circle*, which is the name given to it in standard French works (e.g. *Rouché et Comberousse, Traité de Géométrie.*)

Gaskin (*Construction of a Conic*, 1852, p. 26) calls the circle of Ex. 75 the *director*, because when the conic is a parabola it becomes the directrix as in Ex. 19.

Salmon calls this latter circle the *director circle*, and his example is followed in recent English text-books, though Taylor (*A. and M. Geom. of Conics*) suggests an alternative.

PROBLEM 4.

To draw a pair of tangents to a central conic from an external point.



Let O be the given point.

With centre O and radius OS describe a circle cutting the director circle whose centre is S_1 in M and M_1 (the latter not shown).

Join S_1M and S_1M_1 meeting the curve in P and P_1 (the latter not shown).

Join OP and OP_1; these are the required tangents.

Join OM and OM_1.

$\triangle$s SOP, MOP are congruent; *I.* 8.

$\therefore \quad \angle SPO = \angle MPO$;

$\therefore$ OP is the tangent at P. *Th.* 14.

Similarly, OP_1 is the tangent at P_1.

(Newton, *Principia*, Book I., Lemma 15.)

[The general method of Prob. 2 may also be used.]

Ex. 62. OQ and OQ_1 are a pair of tangents to a central conic; show that SQ, SQ_1, S_1Q, and S_1Q_1 touch a circle whose centre is O.

Ex. 63. A pair of tangents to a central conic make equal angles with the straight lines which join the point from which they are drawn with the foci.

Ex. 64. If an ellipse and a hyperbola have the same foci they cut each other orthogonally.

Ex. 65. If a point move in a plane so that its distance from a given point is equal to its distance from the circumference of a given circle, its locus is a central conic.

Ex. 66. To draw a tangent to a central conic which shall be parallel to a given straight line.

THEOREM 15.

If the tangent and normal at any point P *in a central conic meet the axis in* T *and* G *respectively, and if* PN *and* AN *be the ordinate and abscissa of* P,

i. $SG = eSP$;

ii. $CT \cdot CN = CA^2$;

iii. $NG = \pm(1 - e^2)CN$;

iv. $PN^2 = \pm(1 - e^2)AN \cdot NA_1$.

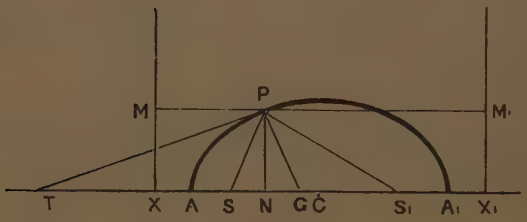


Fig. 1.

From P draw PMM_1 perpendicular to the directrices.

Join S_1P.

i. PG and PT are the internal and external bisectors of the angle SPS_1. *Th.* 14.

$$\therefore \frac{SG}{SP} = \frac{S_1G}{S_1P}; \qquad \textit{VI. 3 or A.}$$

$$\therefore \frac{SG}{SP} = \frac{SG \pm S_1G}{SP \pm S_1P} = \frac{2 \cdot CS}{2 \cdot CA} = e; \qquad \textit{Th. 11, Cor. 2.}$$

$$\therefore SG = eSP.$$

ii. $\dfrac{ST}{S_1T} = \dfrac{SP}{S_1P} = \dfrac{PM}{PM_1} = \dfrac{NX}{NX_1}$; *VI.* 3 *or A, &c.*

$$\therefore \quad \frac{ST + S_1T}{\pm(ST - S_1T)} = \frac{NX + NX_1}{\pm(NX - NX_1)};$$

$$\therefore \quad \frac{2 \cdot CT}{2 \cdot CS} = \frac{2 \cdot CX}{2 \cdot CN};$$

$\therefore$ $CT \cdot CN = CS \cdot CX = CA^2$. *Th.* 11, *Cor.* 2.

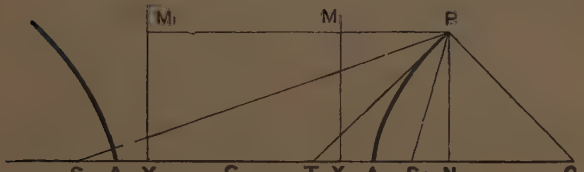


Fig. 2.

iii. $CS = e^2CX$. *Th.* 11, *Cor.* 2.

$SG = eSP = e^2PM = e^2NX$. *Th.* 15, i. *&c.*

$\therefore$ $CG = CS \mp SG = e^2(CX \mp NX) = e^2CN$.

$\therefore$ $NG = \pm(CN - CG) = \pm(1 - e^2)CN$.

iv. Since $\angle TPG$ is a right angle,

$PN^2 = TN \cdot NG$. *VI.* 8.

$= (CT - CN)(1 - e^2)CN$. *Th.* 15, iii.

$= (1 - e^2)(CT \cdot CN - CN^2)$.

$= (1 - e^2)(CA^2 - CN^2)$. *Th.* 15, ii.

$= \pm(1 - e^2)AN \cdot NA_1$. *II.* 5 *or* 6.

Converse of part iv. *If* A *and* A_1 *be fixed points and* PN *a variable ordinate to the straight line* AA_1, *and if* $PN^2 = \pm(1 - e^2)AN \cdot NA_1$, *the locus of* P *is a central conic whose major axis is* AA_1 *and whose eccentricity is e.*

Def. The circle whose diameter is AA_1 is called the *auxiliary circle.*

Cor. 1. If in any ordinate PN of an ellipse there be taken a point Q such that $QN : PN :: CA : CB$ or $(1 - e^2)QN^2 = PN^2$, the locus of Q is the auxiliary circle; and conversely.

For $(1 - e^2)QN^2 = PN^2 = (1 - e^2)AN \cdot NA_1$;

$$\therefore \quad QN^2 = AN \cdot NA_1;$$

$\therefore$ the locus of Q is the auxiliary circle.

Def. P and Q are called *corresponding points.*

Cor. 2. Tangents to an ellipse and its auxiliary circle at corresponding points intersect on the major axis.

Def. The hyperbola whose axes are equal is called the *equilateral hyperbola.*

Cor. 3. If in any ordinate PN of a hyperbola there be taken a point Q such that $QN : PN :: CA : CB$, or $(e^2 - 1)QN^2 = PN^2$, the locus of Q is the equilateral hyperbola which has the same auxiliary circle. Also the tangent to the auxiliary circle from N is equal to QN.

Cor. 4. The latus rectum of a central conic is a third proportional to the axes.

Let SL be the semi-latus-rectum.

$$SL^2 = \pm(1 - e^2)AS \cdot SA_1 = (1 - e^2)(CA^2 - CS^2)$$
$$= \pm(1 - e^2)CB^2.$$

But $CB^2 = \pm (1 - e^2) CA^2$.

$\therefore$ CA : CB : : CB : SL.

Ex. 67. If in a central conic the tangent at P meet the minor axis in t, and if Pn be perpendicular to that axis, $Ct \cdot Cn = CB^2$.

Ex. 68. In the figure of Th. 15, show that

i. S_1SGT is a harmonic range.

ii. $CG \cdot CT = CS^2$.

Ex. 69. If in PN any ordinate of a central conic a point Q be taken such that the ratio PN : QN is constant, the locus of Q is a central conic.

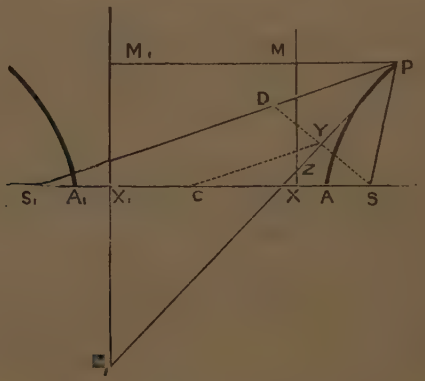
Ex. 70. AA_1 is the diameter of a circle, from B any point in the tangent at A is drawn another tangent BC, BA_1 and CA intersect at P; find the locus of P.

Ex. 71. P and Q are corresponding points in an ellipse and its auxiliary circle; find the locus of the point of intersection of the normal to the ellipse at P and the radius CQ of the circle.

Ex. 72. Find the distance from the centre of a chord of a given ellipse which is parallel to the major axis and equal to half of it.

THEOREM 16.

The image of a focus of a central conic in any tangent lies in the director circle whose centre is the other focus.



[See also figure of Th. 14.]

Let PZ be the tangent at P.

Join SP and S_1P.

Let S_1P meet the director circle in D.

Join SD, meeting PZ in Y.

$\triangle$s SPY, DPY are congruent; *I.* 4.

$\therefore$ SY = YD, and PY perpendicular to SD;

$\therefore$ D is the image of S in PZ;

i.e., the image of S in any tangent is in the director circle whose centre is S_1.

Cor. Y is in the auxiliary circle; or the locus of the foot of a perpendicular from a focus on a tangent is the auxiliary circle.

Join CY.

C is the middle point of SS_1, and Y that of SD;

$\therefore \quad CY = \frac{1}{2}S_1D = \frac{1}{2}AA_1$;

$\therefore$ Y is in the auxiliary circle.

[In the limiting case of the parabola, the auxiliary circle becomes infinite and coincides with the tangent at the vertex. See Th. 6, Cor. 2.]

Ex. 73. Given three tangents to a central conic and one focus, to find the other focus.

Ex. 74. SY and S_1Y_1 are perpendiculars from the foci of a central conic on any tangent; show that $SY \cdot S_1Y_1 = CB^2$.

Ex. 75. If a pair of tangents to a central conic cut each other at right angles, the locus of their point of intersection is a circle.

Ex. 76. An ellipse slides between two straight lines at right angles to each other, find the locus of its centre.

Ex. 77. Given the foci and one tangent to a central conic, to construct the conic.

Ex. 78. If two equal ellipses be placed with their vertices in contact and their axes in opposite directions,

and if one be made to roll on the other, its foci trace the director circles of the other.

Ex. 79. Every point in the director circle of an ellipse is the vertex of a triangle inscribed in the circle and circumscribed about the ellipse.

Def. If from any point in a given figure a perpendicular be drawn to a given plane, the locus of the foot of the perpendicular is called the *orthogonal projection* of the given figure on the given plane.

In what follows *projection* is written for *orthogonal projection.*

Cor. 1. The projections of a straight and of a curved line are in general a straight and a curved line.

Cor. 2. If two figures intersect so do their projections, though not generally at the same angle.

Cor. 3. If two figures touch each other, so do their projections.

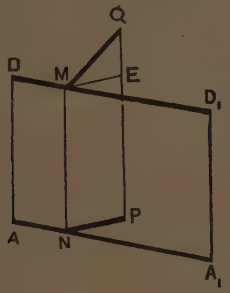
Cor. 4. The projections of parallel straight lines are parallel.

Cor. 5. The length of the projection of a straight line of length l is $l \cos \theta$, where θ is the angle it makes with its projection.

Cor. 6. The ratio of the segments of a straight line is unaltered by projection.

THEOREM 17.

The projection of a circle on any plane is in general an ellipse.



Let DD_1 be that diameter of the circle which is parallel to the plane of projection, and AA_1 its projection.

Take QM any ordinate of the circle, and let PN be its projection.

The locus of P is the projection of the circle.

Draw ME perp. to QP.

Then $AA_1 = DD_1$, PN is perp. to AA_1, $ME = PN$, and $\angle QME = \theta$, the angle between the planes QDD_1 and PAA_1.

$$PN = ME = QM \cos\theta.$$

But $QM^2 = DM \cdot MD_1 = AN \cdot NA_1$; *VI.* 8 *and I.* 34.

$$\therefore \quad PN^2 = QM^2 \cos^2\theta = \cos^2\theta \cdot AN \cdot NA_1;$$

$\therefore$ the locus of P is an ellipse whose major axis is

equal to the diameter of the given circle and whose eccentricity is such that $1 - e^2 = \cos^2\theta$ or $e = \sin\theta$.

Cor. 1. When the planes are parallel the projection is an equal circle.

Cor. 2. A circle can always be found of which a given ellipse is the projection.

THEOREM 18.

The following properties of the circle are true of the ellipse, since the projections of parallel straight lines are parallel, and the ratio of the segments of a straight line is unaltered by projection.

i. *Tangents at the ends of a diameter are parallel.*

ii. *A diameter bisects all chords parallel to the tangents at its ends.*

iii. *Tangents at the ends of a chord intersect on the diameter which bisects the chord.*

Cor. If CP be a semi-diameter which bisects a chord QQ_1 in V and meets the tangent at Q in O, $CV \cdot CO = CP^2$.

Def. Diameters each of which bisects all chords parallel to the other are called *conjugate diameters.* They are the projections of diameters of a circle which are at right angles to each other.

THEOREM 19.

If two chords of an ellipse intersect, the rectangles contained by their segments are to each other as the squares of the semi-diameters to which they are parallel.

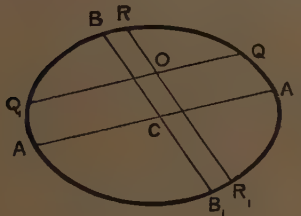


Fig. 1.

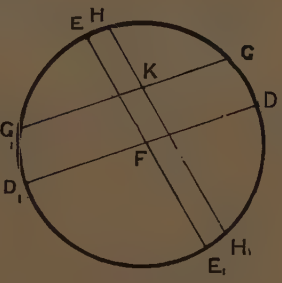


Fig. 2.

Let fig. 1 be the projection of fig. 2, θ the inclination of AA_1 and QQ_1 to DD_1 and GG_1, and ϕ that of BB_1 and RR_1 to EE_1 and HH_1.

Then $OQ \cdot OQ_1 = KG \cdot KG_1 \cdot \cos^2\theta$;

$CA^2 = FD^2 \cos^2\theta$;

$OR \cdot OR_1 = KH \cdot KH_1 \cos^2\theta$;

and $CB^2 = FE^2 \cos^2\phi$ *Cor.* 5, p. 176.

Now $KG \cdot KG_1 = KH \cdot KH_1$, *III.* 35.

and $FD = FE$;

$\therefore \quad OQ \cdot OQ_1 : OR \cdot OR_1 :: CA^2 : CB^2$.

Ex. 80. If an ellipse cut the sides of a triangle, or those sides produced, a in D and D_1, b in E and E_1, and c in F and F_1, show that

$$\frac{BD}{CD} \cdot \frac{CE}{AE} \cdot \frac{AF}{BF} = \frac{CD_1}{BD_1} \cdot \frac{AE_1}{CE_1} \cdot \frac{BF_1}{AF_1}.$$

Ex. 81. Extend Ex. 80 to any polygon cut by an ellipse.

Ex. 82. Using the notation of Ex. 80, show that if AD, BE, and CF are concurrent, so are AD_1, BE_1, and CF_1.

Ex. 83. The straight lines which join the vertices of a triangle with the points of contact of an inscribed ellipse are concurrent.

Ex. 84. The straight lines which join the vertices of a triangle with the points of contact of an escribed ellipse are concurrent.

Ex. 85. If a hexagon be inscribed in an ellipse the points of intersection of its opposite pairs of sides are collinear.

Ex. 86. Given five points in an ellipse, to find any number of points in it.

Ex. 87. ABCDE is a pentagon inscribed in an ellipse, the tangent at A meets CD in X, AB and DE meet in Y, and BC and EA meet in Z; show that X, Y, and Z are collinear.

Ex. 88. If a complete quadrangle be inscribed in an

ellipse, the tangents at any two of its vertices intersect on the sides of its diagonal triangle.

Ex. 89. If a complete quadrilateral be described about an ellipse, it has the same diagonal triangle as the complete quadrangle whose vertices are the points of contact.

Ex. 90. If a triangle be inscribed in an ellipse, and if at the vertices tangents to the ellipse be drawn, the points in which the tangents meet the opposite sides of the triangle are collinear.

Ex. 91. If a circle cut an ellipse in four points, the straight lines which join corresponding pairs of points make equal angles with the axis.

Ex. 92. If an ordinate PN be drawn from any point P in an ellipse to the minor axis, and if in it a point Q be taken such that PN : QN : : CA : CB, the locus of Q is the circle whose diameter is the minor axis.

Ex. 93. Hence any ellipse may be projected into a circle.

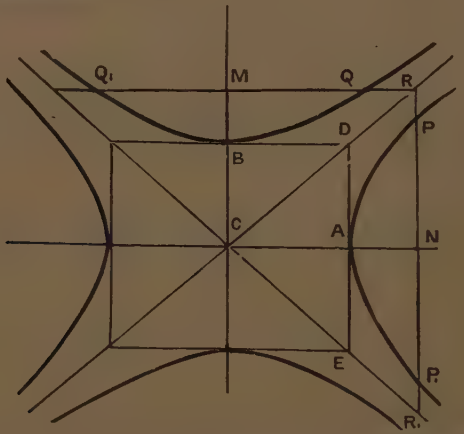
Ex. 94. Hence also a section of a right cylinder by a plane is in general an ellipse.

Ex. 95. Given an ellipse, to find its centre.

Def. If the distance between a straight line and a curve, both of which are indefinitely produced, diminishes without limit, but never vanishes, the straight line is called an *asymptote* of the curve.

THEOREM 20.

The diagonals of the rectangle formed by the tangents at the vertices of a hyperbola and its conjugate are asymptotes of both hyperbolas.



From the symmetry of the figure it is evident that the diagonals pass through C.

In CD take any point R.

Through R draw RN and RM perpendicular to the axes and meeting the curves in P and P_1, Q and Q_1, and CE in R_1.

$\triangle$s RNC, DAC are similar; *VI.* 4.

$\therefore$ $RN : CN = DA : CA = CB : CA$.

But $CB^2 = (e^2 - 1)CA^2$;

$\therefore$ $RN^2 = (e^2 - 1)CN^2$.

Also $PN^2 = (e^2 - 1)\, AN \,.\, NA_1$

$= (e^2 - 1)(CN^2 - CA^2)$. *Th.* 15 *and II.* 6.

Now $RP \cdot RP_1 = RN^2 - PN^2$, *II.* 6.

$= (e^2 - 1)\, CN^2 - (e^2 - 1)\, (CN^2 - CA^2)$,

$= (e^2 - 1)\, CA^2$,

$= CB^2$; *i.e.*, $RP \cdot RP_1$ is constant.

Now as N moves from the vertex, RP_1 increases without limit; $\therefore$ RP diminishes without limit;

$\therefore$ CR is an asymptote of the hyperbola.

Similarly $RQ \cdot RQ_1 = CA^2$;

$\therefore$ CR is an asymptote of the conjugate hyperbola.

In like manner it may be shown that CR_1 is an asymptote of both curves.

Cor. The asymptotes may be regarded as the tangents to the hyperbola from C.

For $CT \cdot CN = CA^2$. *Th.* 15.

But here CT is zero; $\therefore$ CN is positive or negative infinity; $\therefore$ the points of contact are at infinity.

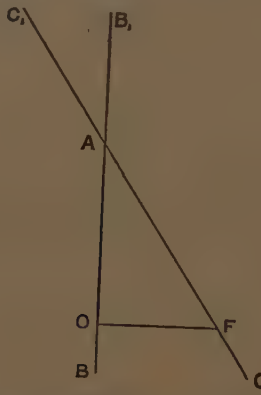
Ex. 96. In the figure of Th. 20, PQ is parallel to one asymptote and is bisected by the other.

Ex. 97. The asymptotes of an equilateral hyperbola cut each other at right angles. (Hence it is also called a *rectangular* hyperbola.)

Ex. 98. If from any point P in an equilateral hyperbola PX and PY be drawn perpendicular to the asymptotes, $PX \cdot PY = \frac{1}{2} CA^2$.

Ex. 99. The straight lines which join the foci with the points in which the asymptotes of a hyperbola meet the directrices are tangents to the auxiliary circle.

Ex. 100. The asymptotes of a hyperbola meet the directrices on the auxiliary circle.



Def. The surface generated by a straight line CAC_1 which cuts a fixed straight line BAB_1 in a fixed point A at a constant angle θ is called a *conical surface* or *cone* (more accurately a *right circular cone*).

BAB_1 is called the *axis*, A the *vertex*, θ the *semi-vertical-angle*, and CAC_1 in any one of its positions a *side* or *generator*.

A conical surface thus consists of two infinite *sheets* on opposite sides of the vertex.

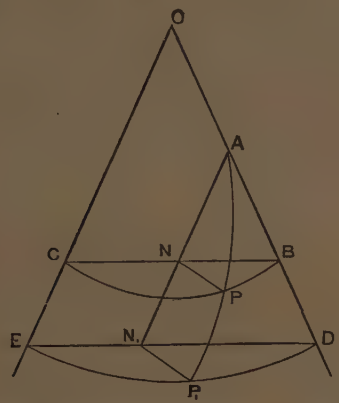
Since $OF = OA \tan\theta =$ a constant (when O is fixed) the section of a cone by a plane through any point O perpendicular to the axis is a circle.

[The solid intercepted between such a circle and the vertex is the solid commonly called a cone.]

The section of a cone by a plane through the axis is obviously a pair of intersecting straight lines.

THEOREM 21.

The section of a cone by a plane parallel to a generator is a parabola.



Let the plane ANP perpendicular to the plane of the paper be parallel to the generator OE; it is required to prove APP_1 a parabola.

Draw any two planes perpendicular to the axis cutting the cone in the circles CPB and EP_1D, and let their diameters cut the plane APP_1 in N and N_1 : join PN, P_1N_1.

Then PN and P_1N_1 are perp. to ANN_1 and to CB and ED respectively. *XI.* 19.

$\therefore$ $PN^2 = BN \cdot NC$ and $P_1N_1^2 = DN_1 \cdot N_1E$; *VI.* 8.

$\therefore$ $PN^2 : P_1N_1^2 :: BN \cdot NC : DN_1 \cdot N_1E$

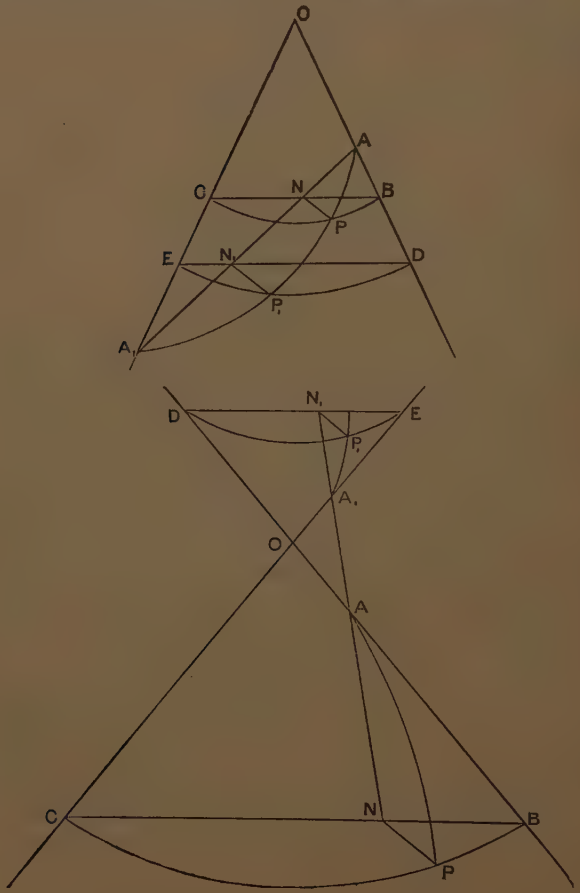
$:: BN : DN_1$ ($\because$ CNN_1E is a parallelogram.)

$:: AN : AN_1$ ($\because$ $\triangle$s ANB, AN_1D sim.)

$\therefore$ APP_1 is a parabola. *Th.* 5, *iv. Conv.*

THEOREM 22.

The section of a cone by a plane which is not parallel to a generator is an ellipse or a hyperbola according as it cuts the cone on the same or on opposite sides of the vertex.



Let the plane ANP perpendicular to the plane of the paper be not parallel to the generator OE but cut it again in A_1, it is required to prove APP_1A_1 an ellipse or a hyperbola.

Draw any two planes perpendicular to the axis cutting the cone in the circles CPB and EP_1D, and let their diameters cut the plane APP_1 in N and N_1 : join PN, P_1N_1.

Then PN and P_1N_1 are perp. to ANN_1 and to CB and ED respectively. *XI.* 19.

$\therefore$ $PN^2 = BN \cdot NC$ and $P_1N_1^2 = DN_1 \cdot N_1E$. *VI.* 8.

$\therefore$ $PN^2 : P_1N_1^2 :: BN \cdot NC : DN_1 \cdot N_1E$.

But $\triangle$ ANB is similar to AN_1D,

and $\triangle$ A_1NC is similar to A_1N_1E. *VI.* 4.

$\therefore$ $BN : DN_1 :: AN : AN_1$.

and $NC : N_1E :: A_1N : A_1N_1$.

$\therefore$ $BN \cdot NC : DN_1 \cdot N_1E :: AN \cdot NA_1 : AN_1 \cdot N_1A_1$.

$\therefore$ $PN^2 : P_1N_1^2 :: AN \cdot NA_1 : AN_1 \cdot N_1A_1$.

$\therefore$ APP_1A_1 is an ellipse or a hyperbola. *Th.* 15, *Cor.*

Cor. When θ is zero the cone becomes a cylinder : hence, an oblique section of a cylinder is an ellipse (Ex. 94.)

Ex. 101. If a sphere be inscribed in the cone in figure of Th. 21 to touch the plane ANP, the point of contact is the focus; and if the plane of contact of the sphere and cone be produced to cut the plane ANP produced, the common section of these planes is the directrix of the parabola APP_1.

Ex. 102. If spheres be inscribed in the cone in figure of Th. 22 to touch the plane ANP, the points of contact are the foci; and if the planes of contact of the spheres and cone be produced to cut the plane ANP produced, the common sections of these planes are the directrices of the ellipse or hyperbola APP_1. (*Quetelet and Dandelin*, 1822.)

QUESTIONS IN GEOMETRICAL CONICS

SET IN THE LEAVING CERTIFICATE EXAMINATION FOR SCOTCH SCHOOLS.

1893.

1. Define a parabola.

If the tangent and ordinate drawn from a point P in a parabola meet the axis in T and N, prove that the subtangent TN is bisected by the vertex of the curve.

If TP meet the directrix and *latus rectum* produced in D and L, show that TD = LP.

2. If the normal at P meet the axis of the parabola in G, prove that the subnormal NG is equal to half the *latus rectum*.

If P'T', P'G' be the tangent and normal of another point, P', on the curve, show that T'T = GG'.

3. AB is the diameter of a circle, and CD is a chord at right angles to AB; show that the parabola which has C for its focus and AD for its directrix, touches AB at B.

4. If QV be an ordinate to a diameter PV of a parabola, then $QV^2 = 4SP \cdot PV$.

5. P is a point on a conic of which S is focus, and Q is a point on the directrix. Find the point in which PQ, produced if necessary, will again meet the curve.

6. Show that the feet of the perpendiculars from the foci of an ellipse on any tangent lie on a fixed circle.

Hence show how to draw two tangents from an external point to a given ellipse.

7. AB, AC are two generating lines of a right circular cone. Prove that any plane parallel to the plane ABC will intersect the cone in a hyperbola. Explain the apparent exception.

1894.

1. The focus and directrix of a parabola being given, find the points in which a given straight line through the focus meets the curve.

2. The tangent to a parabola makes equal angles with the focal radius and the diameter through the point.

A and B are two given points in a circle. Find the focus and directrix of a parabola, which shall touch the circle at A, and whose axis shall touch the circle at B.

3. If PN be an ordinate to a parabola at a point P, prove that $PN^2 = 4AS \cdot AN$.

If AN be produced to K, so that NK is equal to the *latus rectum*, show that APK is a right angle.

4. The *locus* of the foot of the perpendicular from the focus of a parabola on a tangent is the tangent at the vertex.

Show that the circle, described on any focal radius as diameter, touches the tangent at the vertex.

5. If two chords of a parabola intersect, the rectangles contained by their segments are in the ratio of the parameters of the diameters which bisect the chords.

6. If PT, QT are tangents at points P and Q of an ellipse whose foci are S and H, show that the angles STP and HTQ are equal.

Hence show that if an ellipse touch the three sides of a triangle, and one focus be the centre of the circumscribing circle, the other focus will be the orthocentre.

7. If, in a central conic, the tangent at P meet the axis major in T, and N be the foot of the ordinate, show that the semi-axis major is a mean proportional between CT and CN.

8. A sphere rests on a table which is lighted by a bright point whose distance above the table is equal to the diameter of the sphere. Show that the outline of the shadow of the sphere cast on the table is a parabola.

1895.

1. Define a conic section, and show how the definition enables you to distinguish between the three classes of conics.

Explain mechanical methods of describing a parabola and an ellipse.

2. If P be any point of a parabola, S the focus, and Z the point where the tangent at P meets the directrix, prove that PSZ is a right angle.

If A be the vertex, and the tangents at A and P intersect in Y, and YQ be the perpendicular from Y on SZ, show that $YQ = SA$.

3. OP, OQ are tangents to a parabola, and OBV is drawn parallel to the axis meeting the parabola in B and PQ in V, prove that $PV = VQ$ and $OB = BV$.

OP, OQ are met by another tangent in R and S, and the diameters through R and S meet PQ in K and L; show that PQ is equal to twice KL.

4. The rectangles contained by the respective pairs of segments of two intersecting chords of a parabola are proportional to the parameters of the diameters which bisect the chords.

5. Give a construction for finding where a given straight line parallel to the axis cuts a conic whose focus, directrix, and eccentricity are given.

If this line meet the directrix in M and the conic in Q, Q′, and if V be the mid point of QQ′, show that MV is of constant length.

6. If AA′ be the transverse axis of a central conic, and PN be the ordinate drawn to AA′ from a point P on the curve, prove from the elementary definition that PN^2 bears a constant ratio to $AN \cdot NA'$.

7. Show that the feet of the perpendiculars drawn from the foci of an ellipse to any tangent lie on a fixed circle.

8. AB is a fixed straight line, and on AB as chord any circular arc APB is described. If the arc AP be one-third of APB, show that the locus of P is a hyperbola whose focus is A; and find its vertices.

1896.

1. In the parabola prove that $PN^2 = 4AS \cdot AN$.

If the ordinate NP be produced to P′, so that $NP'^2 = 2NP^2$, show that the locus of P′ is a parabola, and find its focus and directrix.

2. Prove that the focal chord which is bisected by a diameter of a parabola, is equal to four times the focal distance of the extremity of that diameter.

Show how to inscribe in a parabola a focal chord of a given length.

3. Draw two tangents OP, OQ to a parabola from an external point O, and show that the angle OPS is equal to the angle QOS.

ABC is a triangle right-angled at A, and S is a point in BC. If points P and Q be taken in AB and AC, so that PSQ is a right angle, show that PQ always touches a fixed parabola, whose focus is S, and find the directrix of this parabola.

4. Prove that in any central conic $CS \cdot CX = CA^2$.

Hence show that any circle which passes through S and X, and the circle which is described on the major axis as diameter, cut each other orthogonally.

5. Prove that in an ellipse $SP : ST = CA : CT$, and show that if the auxiliary circle cut the tangent in Q and Q′, CQ and CQ′ are parallel to SP and S′P.

6. Define a hyperbola and an asymptote, and from your definition prove that the hyperbola has two asymptotes.

Show that each of the asymptotes may be regarded as a diameter which coincides with its conjugate diameter.

7. A chord PP′ of a hyperbola meets the asymptote in Q and Q′; prove that $QP = Q'P'$.

A and B are fixed points on the curve, P is any other point on the curve, PA and PB meet an asymptote in Q and R; show that QR is of constant length.

8. Show that an oblique section of a circular cylinder is an ellipse, whose foci are the points of contact of spheres that are inscribed in the cylinder, and touch the plane of section.

1897.

1. Distinguish between the three classes of conic sections.

If an ellipse, a parabola, and a hyperbola have the same

focus S and the same directrix, and if a line drawn from S meet the ellipse in E, the parabola in P, and the hyperbola in H, show that $SH > SP > SE$.

2. A is the vertex of a parabola, and from P any point in the curve, PM, is drawn at right angles to the directrix, PA produced meets the directrix in K; show that MSK is a right angle.

3. Find the locus of the foot of the perpendicular from the focus on the tangent to the parabola.

Two parabolas have the same focus S, and tangents are drawn to both parabolas at their vertices, meeting in T. Show that a straight line through T perpendicular to ST will touch both parabolas, and find the points of contact.

4. If perpendiculars are drawn from the foci of a central conic to a tangent, the rectangle contained by the perpendiculars is equal to the square on the semi-axis minor.

5. State the fundamental property of conjugate diameters in an ellipse, and prove that supplemental chords are parallel to two conjugate diameters.

From P, any point in an ellipse, PN is drawn perpendicular to a fixed diameter DE. Show that PN is a maximum when CP is conjugate to CD.

6. In the hyperbola the portion of any tangent intercepted between the asymptotes is bisected at the point of contact.

Define a rectangular hyperbola, and prove from the definition that if A, A_1 are the vertices and P any point on the curve, the ordinate PN is a tangent to the circle through A, A_1, and P.

7. If a right cone be cut by a plane which is parallel to a generating line of the cone and at right angles to the plane containing that line and the axis of the cone, the section is a parabola.

Show that the *latus rectum* of a parabola cut from a given cone varies as the distance between the vertex of the parabola and the vertex of the cone.

1898.

1. Define a tangent to a conic.

If in any conic the tangent at P cut the directrix in Z, and S is the focus, PSZ shall be a right angle.

2. Prove that the locus of the mid-points of a series of parallel chords of a parabola is a straight line parallel to the axis.

Prove that the locus of points which divide a series of parallel chords of a parabola in the ratio of 3 : 1 is a parabola, and find its focus.

3. The rectangles contained by the segments of any two intersecting chords of a parabola are proportional to the parameters of the diameters which bisect them.

Hence show that if two chords of a parabola are equally inclined to the axis their extremities are concyclic.

4. If the ordinate PN of an ellipse be produced to meet the auxiliary circle in Q, prove that $NP : NQ = CB : CA$.

Hence find the locus of a fixed point on a straight line of given length sliding between two straight lines at right angles.

5. Define a hyperbola from its focal property, and from your definition deduce the direction in which its infinitely distant points lie.

6. If a pair of tangents be drawn to an ellipse from an external point, they shall subtend equal angles at either focus.

A circle is described through S and H, the foci of an ellipse, meeting the curve in P and Q and the axis minor in T; show that TP and TQ are tangents to the ellipse.

7. A sphere rests on a table and is lighted by a bright point whose distance above the table is greater than the diameter of the sphere. Show that the shadow of the sphere is an ellipse and find its foci.

1899.

1. Define a parabola.

Show from the definition that a parabola is the locus of the centre of a variable circle which touches a given circle and a given straight line.

2. The tangent to a parabola at any point bisects the angle formed by the focal distance and the diameter through the point, and the subtangent is bisected by the vertex of the parabola.

P is a point on a parabola such that $PN = AN$, the tangent at P meets the directrix in Z and the normal at P meets ZS in Q; show that (1) $SP = 2SZ$; (2) PN produced bisects ZQ; (3) Q is a point on the parabola.

3. If TP, TQ be two tangents to a parabola of which S is the focus, prove that the triangles SPT, STQ are similar.

If a third tangent to the parabola meets TP, TQ in L, M, show that the circle described about the triangle SLM passes through T.

4. The square of the ordinate of any point in a central conic bears a constant ratio to the rectangle contained by the intercepts between the foot of the ordinate and the vertices.

If PR be drawn at right angles to AP to meet AA_1 in R, show that NR is to NA_1 in a constant ratio.

5. In the ellipse, if the ordinate and normal meet the major axis in N and G respectively, show that $NG : NC = BC^2 : AC^2$.

If from the centre C of an ellipse a perpendicular be drawn to the tangent at P meeting the ordinate NP pro-

duced in Q, show that the locus of Q is an ellipse similar to the given ellipse.

6. The intercepts on any tangent to a hyperbola between the curve and its asymptotes are equal to one another.

Having given a point P and two straight lines AB, AC, show how to draw the tangent at P to the hyperbola passing through P and having AB, AC for its asymptotes.

7. Show how to find the angle between the asymptotes of a hyperbolic section of a cone.

Hence find under what circumstances it is impossible to obtain a rectangular hyperbola as a section of a given right circular cone.

1900.

1. Define a conic section, its focus, directrix, and eccentricity.

If PQ be a chord of a conic section which meets the directrix in R, and S be the focus, prove that SR bisects the angle PSQ either internally or externally, illustrating both cases by figures.

2. If PN be the ordinate and PG the normal at a point P of a parabola, prove that $SP = SG$ and $NG = 2AS$.

PP_1 is a focal chord which makes an angle of 60° with the axis of a parabola; prove that it is divided by the focus into segments which are in the ratio of 1 to 3.

3. If in a parabola QV be an ordinate to a diameter PV, then $QV^2 = 4SP \cdot PV$.

AB is a fixed line and C a fixed point outside AB; in AB any point R is taken, and RV is drawn to meet AC in V so that the angle ARV is equal to ACR; the parallelogram RAVQ being completed, find the locus of Q.

4. Prove that in a central conic the tangent makes equal angles with the focal distances.

An ellipse and hyperbola have the same foci, prove that they intersect at right angles.

5. If SY, HZ are the perpendiculars from the foci S, H of an ellipse on a tangent, then the rectangle contained by SY, HZ is constant.

The locus of the intersection of tangents to an ellipse which are at right angles to each other is a fixed circle.

6. If from any point P on a hyperbola, lines PQ, PR be drawn parallel to the asymptotes to meet them in Q, R, the rectangle PQ . PR is constant.

If a straight line through the centre meet PQ, PR in S, T, and the parallelogram TP . SU be completed, show that U will be a point on the hyperbola.

7. If a plane be parallel to one and one only of the generating lines of a cone, its intersection with the cone will be parabola.

Find the locus of the vertices of all the parabolas which can be cut from the cone and which have the same latus rectum as the given parabola.

1901.

1. If P be a point on a conic, PM the perpendicular on the directrix, PG the normal at P meeting the axis in G, prove that $PG = eSM$ where S is the focus, and e the eccentricity.

2. Prove that the tangents drawn to a parabola at the extremities of a focal chord intersect at right angles in the directrix.

If the normal at L, the extremity of the latus rectum, meet the axis in M and the curve in N, prove that $MN = 3LM$.

3. Prove that the rectangles contained by the segments of two intersecting chords of a parabola are in the ratio of the parameters of the diameters which bisect the chords.

If the extremities of the chords lie on a circle, show that the chords are equally inclined to the axis of the parabola.

4. If PN be the perpendicular drawn to the axis from any point of an ellipse, prove that

$$PN^2 : AN \cdot NA_1 :: BC^2 : AC^2.$$

Deduce the relation between any ordinate of an ellipse and the ordinate of the corresponding point of the auxiliary circle.

5. If from a point O, tangents OP, OQ be drawn to an ellipse of which S is a focus, prove that the angles OSP, OSQ are equal.

If a tangent to an ellipse meet the tangents at its vertices in R, R_1, prove that the circle on RR_1 as diameter passes through both foci.

6. Define an asymptote.

If a chord of a hyperbola meet the curve in P, P_1 and the asymptotes in Q, Q_1, prove that $QP = P_1Q_1$.

7. If a plane cut a right cone so as to meet all the generating lines on the same side of the vertex, prove that the curve of section is an ellipse.

If A, B be the vertices of the ellipse, and V the vertex of the cone, show that the eccentricity of the ellipse is $\frac{VA - VB}{AB}$.

1902.

1. Prove that in any conic the part of the tangent intercepted between the curve and the directrix subtends a right angle at the focus.

PQ is a tangent to a conic at P which meets the directrix at Q, and R is another point in the directrix; if P, Q, R and the eccentricity of the conic be known, find the focus.

2. Prove that in the parabola $PN^2 = 4AS \cdot AN$ PP_1 is a chord of a parabola which makes equal angles with the axis and directrix; show that the latus rectum is equal to the sum or difference of the ordinates of P and P_1 according as these points are on the same or opposite sides of the axis.

3. Prove that any diameter of a parabola bisects all chords parallel to the tangent at its extremity.

4. If the tangent to the ellipse at P meet the axis minor BC in t, and n be the foot of the perpendicular from P on BC, prove that $BC^2 = Ct \cdot Cn$.

Two ellipses have the same centre and vertices; prove that the tangents at the extremities of their latera recta meet on the axis minor produced.

5. In the ellipse prove that the sum of the squares on two conjugate diameters is constant.

6. Prove that any tangent to a hyperbola makes with the asymptotes a triangle of constant area.

A is the vertex of an isosceles triangle ABC, P is any point in AB, and BQ is drawn parallel to PC to meet AC produced in Q; prove that PQ always touches a certain hyperbola, and find the vertex and focus.

1903.

1. Define a parabola, its focus, directrix, and axis.

Prove that in a parabola the subtangent is equal to twice the abscissa, and that the subnormal is constant.

2. If in a parabola SY be the perpendicular from S on the tangent at P, prove that Y lies on the tangent at the vertex A, and that SY is a mean proportional between SP and SA.

Prove that PY is a mean proportional between SP and the abscissa AN.

3. If QV be an ordinate to the diameter PV of a parabola, prove that $QV^2 = 4SP \cdot PV$.

4. If S, S_1 be the foci of a central conic, A, A_1 the vertices, and X, X_1 the points in which the transverse axis meets the directrices, prove that AA_1 is a mean proportional between SS_1 and XX_1.

Prove that any circle through S and X is cut orthogonally by the circle described on AA_1 as diameter.

5. If from the foci of a central conic perpendiculars are drawn to the tangent at any point, prove that the feet of the perpendiculars are on the circumference of the circle described on the transverse axis as diameter.

If A be a point within a circle, C the centre, and P

any point in the circumference, prove that the straight line which bisects AP at right angles is a tangent to an ellipse whose foci are A and C.

6. Prove that if all the generating lines of a cone be produced beyond the vertex, and a plane cut both the cone and the produced cone, the section will be a hyperbola.

INDEX

	PAGE
Anharmonic Ratio	49
Brianchon's Theorem	97
Ceva's Theorem	12, 14
Co-axal Circles	61
Complete Quadrangle	23
Complete Quadrilateral	23
Cross Ratios	49
Desargues' Theorem	17
Ellipse and Hyperbola	161
General Conic	135
Harmonic Ranges and Pencils	23
Inversion	82
Involution	57
Linkages	100
Menelaus' Theorem	9, 14
Orthogonal Projection	176
Pantagraph	100
Parabola	145
Pascal's Theorem	20
Peaucellier Cell	102
Pole and Polar	40
Principle of Continuity	ix
Principle of Duality	x
Ptolemy's Theorem	94

PAGE

Radical Axes 61
Ranges and Pencils 1
Reciprocal Polars 96
Similitude 74
Transversals 1

Definition of—
Abscissa 136
Anharmonic Ratio of a Pencil 49
Anharmonic Ratio of a Range 49
Antiparallels 86
Asymptote 181
Auxiliary Circles 172
Axis of Similitude 79
Centre of Inversion 82
Centre of Involution 57
Centre of Similitude 74
Co-axal Circles 70
Collinear 1
Complete Quadrangle 33
Complete Quadrilateral 33
Concurrent 2
Cone 184
Conic Section 135
Conjugate Diameters 178
Conjugate Hyperbolas 163
Constant of Inversion 82
Constant of Involution 57
Cross Ratio of a Pencil 49
Cross Ratio of a Range 49

Definition of— PAGE
Diagonal Triangle 33
Diameter of a Central Conic 162
Diameter of a Parabola 154
Director Circle 167
Directrix 135
Double Points 57
Double Rays 58
Eccentricity 135
Ellipse 135
Equilateral Hyperbola 172
Focus 135
Harmonic Conjugates 23
Harmonic Division 23
Harmonic Mean 23
Harmonic Pencil 27, 28
Harmonic Progression 23
Harmonic Range 23
Homothetic Figures 74
Hyperbola 135
Inverse Points 82
Involution of Six Points 57
Involution of Six Rays 58
Isogonal Conjugates 16
Latus Rectum 136
Limiting Points 72
Linkage 100
Normal 130
Ordinate 136
Orthogonal Circles 65
Orthogonal Projection 176

Definition of— PAGE
Parabola 135
Parameter 136
Pencil 2
Polar of a Point with respect to a Circle . . 41
Polar of a Point with respect to an Angle . . 32
Pole of a Straight Line with respect to a Circle . 41
Position Ratio of a Point 3
Position Ratio of a Ray 5
Potency of a Point with respect to a Circle . . 61
Radical Axis 63
Radical Centre 65
Range 1
Ratio of Similitude 74
Ray 2
Rectangular Hyperbola 183
Subnormal 136
Subtangent 136
Symmedian 16
Transversal 7

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